Module: 203-02

Even semester-2020

Lecture note on Identical Particles in Quantum Mechanics

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Introduction:

Let us consider two particles collides each other and we are trying to detect them before and after collision. What would happen?

1. If these two particles were simple bowling balls having different colors or different masses we do not have any trouble in identifying the path of each ball before, during and after collision.
2. Even if the balls were completely identical we could still track their separate trajectories as long as we can know the position and momentum of each balls simultaneously.
3. Now let us thinking about the collisions between two protons or two neutrons or two electros i.e the collision between two identical sub-atomic particles.
4. These particles are perfectly identical in every way.
5. According to uncertainty principle one cannot track the exact position and momentum of each particle.
6. We could not able to detect which particles enters detector after collision.
7. Finite extent of each particle wave functions leads to overlapping wave functions around collision region and we can't know which wave function belongs to which particle.

By identical particles we mean that the particles which cannot be distinguished by means of their inherent properties like mass, spin etc. In general identical particles are broadly categorized in two groups:

(i) Classical particles. These particles are identical but distinguishable due to their classical nature. Example: two identical balls of same colour.

(ii) Quantum particles. These particles are identical but indistinguishable due to their quantum mechanical nature. Example: Electrons, Protons, Neutrons etc.

The Spin-Statistics Theorem: Identical indistinguishable particles are again broadly categorized in two groups on the basis of their intrinsic spin.

Category-I: Bosons

Systems of identical particles with integer spin (s = 0, 1, 2,--------), known as bosons. The total wave function of the system of bosons is symmetric under interchange of any pair of particle positions. The wave function is said to obey Bose-Einstein statistics.

Category-II: Fermions

Systems of identical particles with half-odd-integer spin (s = 1/2, 3/2,-------), known as fermions. The total wave function of the system of bosons is anti-symmetric under interchange of any pair of particle positions. The wave function is said to obey Fermi-Dirac statistics.

When particles are indistinguishable:

Identical particles also called indistinguishable particles are particles that cannot be distinguished from one another. Identical or indistinguishable particles behaviors can be studied using quantum mechanical approach as explained below.

Let us consider a system of two electrons. The Hamiltonian for the system of two electrons leveled as 1 and 2 is given by

$$\hat{H}(1,2) = \frac{\hat{p}_1{}^2}{2m_e} + \frac{\hat{p}_2{}^2}{2m_e} + \hat{V}(\vec{r}_1, \vec{r}_2) \tag{1}$$

where m_e is the mass of each electron. As the particles are indistinguishable Hamiltonian of the system must be invariant under particle exchange i.e,

$$\hat{H}(1,2) = \hat{H}(2,1) \tag{2}$$

If equation (2) does not hold we would get differences in the measurement and that differences put us in violation of uncertainty principle. For the system of identical particles the energy remains same with particle exchange: i.e.

$$\hat{H}(1,2)\, \Psi(\vec{x}_1, \vec{x}_2) = \mathrm{E}\, \Psi(\vec{x}_1, \vec{x}_2) \tag{3}$$

and

$$\hat{H}(2,1)\, \Psi(\vec{x}_2, \vec{x}_1) = \mathrm{E}\, \Psi(\vec{x}_2, \vec{x}_1) \tag{4}$$

Equality of $\hat{H}$ (1,2) and $\hat{H}$ (2,1) does not imply that $\Psi(\vec{x}_1, \vec{x}_2)$ and $\Psi(\vec{x}_1, \vec{x}_2)$ are equal.

According to quantum mechanics:

1. $|\Psi\ (\vec{x}_1, \vec{x}_2)|^2$ is the probability density of the state for particle 1 to be at $\vec{x}_1$ when particle 2 to be at $\vec{x}_2$.
2. Similarly $|\Psi\ (\vec{x}_2, \vec{x}_1)|^2$ is the probability density of the state for particle 1 to be at $\vec{x}_2$ when particle 2 to be at $\vec{x}_1$.
3. These two probabilities are not necessarily same.

But we require that probability density do not depend on how we label particles.

Since $\hat{H}(1, 2) = \hat{H}(2,1)$, the following equations must hold that

$$\hat{H}(1, 2)\, \Psi(\vec{x}_1, \vec{x}_2) = \hat{H}(2,1)\, \Psi(\vec{x}_1, \vec{x}_2) = \mathrm{E}\, \Psi(\vec{x}_1, \vec{x}_2) \tag{5}$$

and similarly

$$\hat{H}(1, 2)\, \Psi(\vec{x}_2, \vec{x}_1) = \hat{H}(2,1)\, \Psi(\vec{x}_2, \vec{x}_1) = \mathrm{E}\, \Psi(\vec{x}_2, \vec{x}_1) \tag{6}$$

Both Ψ $(\vec{x}_1, \vec{x}_2)$ and Ψ $(\vec{x}_2, \vec{x}_1)$ share same energy Eigen value E, so any linear combination of $\Psi(\vec{x}_1, \vec{x}_2)$ and $\Psi(\vec{x}_2, \vec{x}_1)$ will be a Eigen state of $\hat{H}$ having same energy eigen value E. However a linear combination of $\Psi(\vec{x}_1, \vec{x}_2)$ and $\Psi(\vec{x}_2, \vec{x}_1)$ may or may not preserves indistinguishability.

Particle Exchange Operator:

The particle exchange operator denoted as $\hat{P}_{12}$is defined by the following relation

$$\hat{P}_{12}\, \Psi\, (r_1, s_1;\ r_2, s_2) = \Psi\, (r_2, s_2; r_1, s_1) \qquad (1)$$

Where r_1 *and* r_2 are the position vector of particle 1 and 2 respectively and s_1 *and* s_2 are their respective spin vectors. The function of the particle exchange operator is to interchange the subscripts of the positions and spins for particles 1 and 2 of the composite wave function. If the two particles are truly identical then the Hamiltonian of the system must be symmetric with respect to the position and spin of the individual particles.

Eigen functions and energy eigen value of the Particle Exchange Operator:

Let 'k' be the eigen value of the particle exchange operator $\hat{P}_{12}$ in the state $\Psi(1,2)$. Here $\Psi(1,2)$ is the state of the system of particles 1 and 2. Therefore the eigen value equation for the particle exchange operator $\hat{P}_{12}$ is

$$\hat{P}_{12}\, \Psi(1,2) = \mathrm{k}\, \Psi(1,2) \qquad (1)$$

Operating equation (1) by $\hat{P}_{12}$ again we get

$$\hat{P}_{12}{}^2\, \Psi(1,2) = \hat{P}_{12}\, \hat{P}_{12}\, \Psi(1,2) = \hat{P}_{12}\, \mathrm{k}\, \Psi(1,2)$$

$$= \mathrm{k}\hat{P}_{12}\Psi(1,2) = \mathrm{k\, k}\, \Psi(1,2) = \mathrm{k}^2\, \Psi(1,2)$$

$$\hat{P}_{12}{}^2\, \Psi(1,2) = \mathrm{k}^2\, \Psi(1,2) \qquad (2)$$

From the definition of particle exchange operator we have

$$\hat{P}_{12}\, \Psi(1,2) = \Psi(2,1)$$

Operating again by $\hat{P}_{12}$we get

$$\hat{P}_{12}{}^2\, \Psi(1,2) = \hat{P}_{12}\, \hat{P}_{12}\, \Psi(1,2) = \hat{P}_{12}\, \Psi(2,1) = \Psi(1,2) \qquad (3)$$

From equations (2) and (3) we get

$$\mathrm{k}^2\, \Psi(1,2) = \Psi(1,2)$$

$$\text{or } \mathrm{k}^2 = 1$$

$$\text{or } k = \pm 1$$

Thus the eigen value of the particle exchange operator are ± 1.

Symmetric and anti-symmetric wave function:

Let us consider a system of n identical indistinguishable particles. The wave function of the system consisting of n particles is Ψ(1,2,3,4……….n, t). The Schrödinger equation for the above system of particles is written as

$$\hat{H}(1,2,3\ldots\ldots\ldots\ldots\ldots n)\,\Psi(1,2,3\ldots\ldots\ldots\ldots n, t) = i\hbar\frac{\partial}{\partial t}\,\Psi(1,2,3\ldots\ldots\ldots\ldots\ldots, t) \qquad (1)$$

where each of the numbers represents all the position and spin coordinates of one of the particles.

As the particles are identical, Hamiltonian $\hat{H}$ of the system is symmetrical in its arguments. Two types of solutions of equation (1) are possible for the wave function of Ψ; namely (i) Symmetric wave function and (ii) anti-symmetric wave function.

(i) **Symmetric wave function (Ψ_S)**
(ii) **Anti-symmetric wave function (Ψ_A)**

Symmetric wave function (Ψ_S): A wave function is said to be symmetric if the interchange between any pair of particles among its arguments do not change the sign of the wave function.

Anti-symmetric wave function (Ψ_A): A wave function is said to be anti-symmetric if the interchange between any pair of particles among its arguments change the sign of the wave function. This may be seen as follows:

Let $\Psi_S = \Psi(1,2) + \Psi(2,1)$ and $\Psi_A = \Psi(1,2) - \Psi(2,1)$.

Therefore

$$\hat{P}_{12}\,\Psi_S = \hat{P}_{12}\,[\Psi(1,2) + \Psi(2,1)] = \hat{P}_{12}\,\Psi(1,2) + \hat{P}_{12}\Psi(2,1)$$

$$= \Psi(2,1) + \Psi(1,2) = \Psi(1,2) + \Psi(2,1) = \Psi_S$$

$$\text{Or, } \hat{P}_{12}\,\Psi_S = \Psi_S$$

Similarly

$$\hat{P}_{12}\,\Psi_A = \hat{P}_{12}\,[\Psi(1,2) - \Psi(2,1)] = \hat{P}_{12}\,\Psi(1,2) - \hat{P}_{12}\Psi(2,1)$$

$$= \Psi(2,1) - \Psi(1,2) = -\,[\Psi(1,2) - \Psi(2,1)] = -\,\Psi_A$$

$$\text{Or, } \hat{P}_{12}\,\Psi_A = -\,\Psi_A$$

If we apply such an interchange operator ($\hat{P}_{12}$) twice on the wave function of the system consisting two particles, brings back to their original configuration and hence produces no change in the wave function.

Commutation relation of $\hat{\boldsymbol{P}}_{12}$ with $\hat{\boldsymbol{H}}$:

We know that

$$\hat{P}_{12}\,\Psi(1,2) = \Psi(2,1)$$

$$\text{Therefore } \hat{P}_{12}\,\hat{H}(1,2)\,\Psi(1,2) = \hat{H}(2,1)\,\Psi(2,1) = \hat{H}(1,2)\,\Psi(2,1) \qquad \text{As } \hat{H}(1,2) = \hat{H}(2,1)$$

$$= \hat{H}(1,2)\,\hat{P}_{12}\,\Psi(1,2)$$

$$\text{Or } [\hat{P}_{12}\,\hat{H}(1,2) - \hat{H}(1,2)\,\hat{P}_{12}]\,\Psi(1,2) = 0$$

$$\text{Or } [\hat{P}_{12}\,\hat{H}(1,2) - \hat{H}(1,2)\,\hat{P}_{12}] = 0$$

$$\text{Or } [\hat{P}_{12}\,,\,\hat{H}(1,2)] = 0$$

Thus the particle exchange operator commutes with the Hamiltonian of the system.