Meromorphic Function

Defn A fn whose only singularities in a region R are poles, is called a meromorphic function in R.

A fn whose only singularities in the open plane are all poles is said to be meromorphic in the plane; if in addition f is analytic or has a pole at infinity then f is said to be meromorphic in the closed plane or extended plane.

For example a rational function, $\tan z$, $\cot z$, $\frac{1}{e^z - 1}$ etc are meromorphic functions.

* A meromorphic fn in the open plane may have infinite number of poles or zeros but there is only a finite number of them in any finite domain otherwise there would exist at least one limit point of poles or zeros in the finite plane and would be an essential singularity. Two cases may be distinguished.

(1) The pt. at infinity is a regular pt or a pole so the fn has a finite number of poles in this case.

(2) The pt. at infinity is an essential singularity. The fn has an infinity number of poles or zeros which accumulate at the pt. at infinity.

the place of rational functions in the class of meromorphic f^ns is shown in the following theorem.

<u>Theorem</u> A rational f^n has no singularities other than poles. Conversely a single valued analytic f^n which has no other singularities than poles at any point (including the pt. at infinity) is a rational f^n or a f^n which is meromorphic in the extended plane is a rational function.

<u>Proof</u> Let f be a rational f^n. Then $f(z) = \frac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are polynomials in z.

Since a polynomial is an integral f^n having a finite number of zeros, f has only a finite no. of poles in the finite z-plane. Also at the point at infinity f may have a regular pt. or a pole.

Conversely let f be a f^n which have no singularities other than poles at any pt. (including the pt. at infinity).

Since f is assumed to have at most a pole at $z=\infty$, it is regular everywhere outside a sufficiently large circle i.e, in a certain nbd. of the pt. $z=\infty$ itself. Hence all singularities except possibly at $z=\infty$ lie in the finite part of the plane points which lie in the finite part of the plane lie within an assignable circle. These can only be a finite number of such points because otherwise there would be a limit point of these singular points in this closed circle. This point is an essential singularity.

(If there is no singular point in the finite part of the plane then f is a polynomial.)

Suppose then that the poles of f at finite points are $\alpha_1, \alpha_2, \ldots, \alpha_k$ and let principal part of its expansion at the point α_r be

$$\frac{a_{r,1}}{z-\alpha_r} + \frac{a_{r,2}}{(z-\alpha_r)^2} + \cdots + \frac{a_{r,n_r}}{(z-\alpha_r)^{n_r}}.$$

Let the principal part of its expansion at the pt. at infinity be $a_1 z + a_2 z^2 + \cdots + a_n z^n$. If there is no pole at the pt. at infinity then all a_r [illegible] $(r = 1, 2, \ldots, n)$.

Now consider the function

$$g(z) = f(z) - \sum_{r=1}^{k} \left\{ \frac{a_{r,1}}{z-\alpha_r} + \frac{a_{r,2}}{(z-\alpha_r)^2} + \cdots + \frac{a_{r,n_r}}{(z-\alpha_r)^{n_r}} \right\}$$
$$- (a_1 z + a_2 z^2 + \cdots + a_n z^n).$$

The function g is evidently regular everywhere in the whole plane including the pt. at infinity and, therefore, is a constant c, say. Hence

$$f(z) = c + \sum_{r=1}^{k} \left\{ \frac{a_{r,1}}{z-\alpha_r} + \frac{a_{r,2}}{(z-\alpha_r)^2} + \cdots + \frac{a_{r,n_r}}{(z-\alpha_r)^{n_r}} \right\}$$

$+ (a_1 z + a_2 z^2 + \cdots + a_n z^n)$,

Which is a rational f^z of z.

<u>Corollary</u> A meromorphic f^z other than a rational f^z must have an essential singularity at the point at infinity. These are called transcendental meromorphic functions.

The functions $\tan z$, $\cot z$, $\sec z$, $\frac{1}{e^z-1}$ etc. are transcendental meromorphic functions.

Nonpolynomial entire functions are called transcendental entire functions.

The functions $\sin z$, $\cos z$, e^z etc. are transcendental entire functions.

<u>Corollary</u> Any rational function possesses a decomposition in partial fractions.

<u>Expansion of a meromorphic function</u>

We know that a rational m.f. can be expressed as a sum of partial fractions. We shall ~~show~~ now obtain a similar expression for a more general class of meromorphic functions.

To this end we prove the following Theorem.

<u>Theorem</u> Let f be a f^z whose only singularities in the finite plane are poles. We shall suppose for simplicity that all these poles are simple. Let them be $a_1, a_2, \ldots$ where $0 < |a_1| \le |a_2| \le \cdots$ and let the residue of f at the pole a_r be b_r. Suppose further that there is a sequence of closed contours C_n around the origin such that (i) C_n includes $a_1, a_2, \ldots, a_n$ but no other poles. (ii) We also suppose that (ii) if R_n is the minimum distance of C_n from the origin, then $R_n \to \infty$ as $n \to \infty$ and (iii) if L_n denotes the length of C_n, then $L_n = O(R_n)$ as $n \to \infty$. i.e. $L_n = \lambda R_n$ when $n \to \infty$.

Regarding f we suppose that (iv) on C_n $|f(z)| = o(R_n)$ i.e, on C_n $\frac{|f(z)|}{R_n} \to 0$ as $n \to \infty$.

(This last condition will be satisfied if, for example f is bounded on the system of contours C_n taken as a whole i.e, $|f(z)| < M$ on C_n).

Under these conditions $f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z-a_n} + \frac{1}{a_n}\right)$ for all z except for poles.

<u>Proof</u> Consider the integral

$$I_n = \frac{1}{2\pi i} \oint_{C_n} \frac{f(w)\,dw}{w(w-z)}, \text{ where } z \text{ is any pt.}$$

within C_n, not coinciding with any pole. We also suppose that $z \neq 0$ because for $z = 0$ the expansion obviously holds.

The integrand $\frac{f(\omega)}{\omega(\omega - z)}$ is regular for all ω, excepting the points $a_1, a_2, \ldots, a_n, 0, z$ where it has simple poles.

The residue at a_m $(m = 1, 2, \ldots, n)$

$$= \lim_{\omega \to a_m} (\omega - a_m) \frac{f(\omega)}{\omega(\omega - z)} = \frac{b_m}{a_m(a_m - z)},$$

since $\lim_{\omega \to a_m} (\omega - a_m) f(\omega) = b_m$.

The residue at $\omega = z$ is

$$\lim_{\omega \to z} (\omega - z) \frac{f(\omega)}{\omega(\omega - z)} = \frac{f(z)}{z}.$$

and the residue at $\omega = 0$ is

$$\lim_{\omega \to 0} \omega \frac{f(\omega)}{\omega(\omega - z)} = -\frac{f(0)}{z}.$$

In particular, when $f(z) = 0$ and $f(0) = 0$ the last two residues vanish.

So by Cauchy's ~~residue~~ residue theorem we get,

$$I_n = \frac{2\pi i}{2\pi i} \left[\text{sum of the residues of } \frac{f(\omega)}{\omega(\omega - z)} \text{ at } a_1, a_2, \ldots, a_n, 0, z\right].$$

$$= \sum_{m=1}^{n} \frac{b_m}{a_m(a_m - z)} - \frac{f(0)}{z} + \frac{f(z)}{z} \quad \cdots (1)$$

On the other hand, on C_n

$$\left| \frac{f(\omega)}{\omega(\omega - z)} \right| \leq \frac{\max |f(\omega)| \text{ on } C_n}{R_n(R_n - |z|)}$$

So by ML-formula we get.

$$|I_n| \leq \frac{L_n}{2\pi R_n(R_n - |z|)} \cdot \max_{\omega \in C_n} |f(\omega)|$$

$$= \frac{1}{2\pi} \frac{L_n}{R_n} \frac{\max_{\omega \in C_n} |f(\omega)|}{\left(1 - \frac{|z|}{R_n}\right) R_n} \to 0 \text{ as } n \to \infty$$

by the given conditions.

Hence $\lim_{n \to \infty} I_n = 0$ and we get from (1) that

$$\sum_{m=1}^{\infty} \frac{b_m}{b_m(a_m - z)} - \frac{f(0)}{z} + \frac{f(z)}{z} = 0$$

i.e, $$f(z) = f(0) - \sum_{m=1}^{\infty} \frac{b_m z}{a_m(a_m - z)}$$

$$= f(0) + \sum_{m=1}^{\infty} b_m \left(\frac{1}{z - a_m} + \frac{1}{a_m} \right),$$

for all z except for poles. This proves the th

~~Poisson~~ Jensen's Formula (P. J Formula)

Theorem Let f have zeros at the points $a_1, a_2, \ldots, a_m$ and poles at $b_1, b_2, \ldots, b_n$ inside the disc $|z| \le R$ and be analytic elsewhere inside and on the boundary of the disc. If $z = re^{i\theta}$ $(0 \le r < R)$ and z does not coincide with any zero or pole then

$$\log|f(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \phi) + r^2} \log|f(Re^{i\phi})|\, d\phi$$

$$- \sum_{\mu=1}^{m} p_\mu \log\left|\frac{R^2 - \bar{a}_\mu re^{i\theta}}{R(re^{i\theta} - a_\mu)}\right| + \sum_{\nu=1}^{n} q_\nu \log\left|\frac{R^2 - \bar{b}_\nu re^{i\theta}}{R(re^{i\theta} - b_\nu)}\right| \quad \text{------ (1)}$$

where p_μ is the order of a_μ and q_ν is the order of b_ν.

Proof Case I Let f be analytic and have no zero and pole in $|z| \le R$. Then each branch of $\log f(z)$ is analytic ~~analytic~~ for $|z| \le R$ and hence by Poisson's integral formula

$$Re\{\log f(re^{i\theta})\} = \log|f(re^{i\theta})|$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\phi})|}{R^2 - 2rR\cos(\theta - \phi) + r^2}\, d\phi,$$

$$= \frac{1}{2\pi}\int_0^{2\pi} P \log|f(Re^{i\phi})|\, d\phi,$$ where P is the Poisson Kernel and we use this symbol throughout the proof. This proves the theorem in the special case when f is analytic and has neither a zero nor a pole in $|z| \le R$.

Case II Let $f(z) = z - a$, $|a| < R$

Then we have to prove that

$$\log|re^{i\theta} - a| = \frac{1}{2\pi}\int_0^{2\pi} P \log|Re^{i\phi} - a|\, d\phi - \log\left|\frac{R^2 - \bar{a} re^{i\theta}}{R(re^{i\theta} - a)}\right|$$

i.e., $$\log\left|R - \frac{\bar{a} re^{i\theta}}{R}\right| = \frac{1}{2\pi}\int_0^{2\pi} P \log|Re^{i\phi} - a|\, d\phi.$$

This is equivalent to Poisson's formula for the real part of the logarithm of the function ~~f(z)~~

$F(z) = \left(R - \frac{\bar{a}z}{R}\right)$ which is analytic for $|z| \le R$ (because the point $R^2/\bar{a}$ is outside the circle $|z| = R$). For, in this case

$$\log|F(re^{i\theta})| = \log\left|R - \frac{\bar{a} re^{i\theta}}{R}\right|$$ and

$$\log|F(Re^{i\phi})| = \log|R - \bar{a}e^{i\phi}|$$

$$= \log \left\{ \frac{|R - \bar{a}e^{i\varphi}|}{|Re^{i\varphi} - a|} \cdot |Re^{i\varphi} - a| \right\}$$

$$= \log |Re^{i\varphi} - a| \quad \left[\because \left| \frac{R - \bar{a}e^{i\varphi}}{Re^{i\varphi} - a} \right| = 1 \right]$$

This proves case II.

<u>Case III</u> Let $f(z) = \frac{1}{z-b}$ $(|b| < R)$.

In this case we are to prove that

$$\log \left| \frac{1}{re^{i\theta} - b} \right| = \frac{1}{2\pi} \int_0^{2\pi} P \log \left| \frac{1}{Re^{i\varphi} - b} \right| d\varphi$$

$$+ \log \left| \frac{R^2 - \bar{b}re^{i\theta}}{R(re^{i\theta} - b)} \right|$$

or. $$\log \left| R - \frac{\bar{b}re^{i\theta}}{R} \right| = -\frac{1}{2\pi} \int_0^{2\pi} P \log \left| \frac{1}{Re^{i\varphi} - b} \right| d\varphi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} P \log |Re^{i\varphi} - b| \, d\varphi .$$

This is equivalent to Poisson's integral formula for the real part of the function $\log\left(R - \frac{\bar{b}z}{R}\right)$, which is analytic for $|z| \le R$.

<u>Case IV</u> In this general case, let

$$f(z) = \frac{(z-a_1)^{p_1} (z-a_2)^{p_2} \cdots (z-a_m)^{p_m}}{(z-b_1)^{q_1} (z-b_2)^{q_2} \cdots (z-b_n)^{q_n}} T(z) \, ; \quad \text{---- (2)}$$

where T is regular and $T(z) \neq 0$, within and on [illegible] To obtain the result in the general [illegible] we shall have to apply [illegible] previous cases.

Now from (2) we get

$$\log |f(re^{i\theta})| = \sum_{\mu=1}^{m} p_\mu \log |re^{i\theta} - a_\mu| + \sum_{\nu=1}^{n} q_\nu \log \frac{1}{|re^{i\theta} - b_\nu|}$$

$$+ \log |T(re^{i\theta})| .$$

$$= \sum_{\mu=1}^{m} p_\mu \frac{1}{2\pi} \int_0^{2\pi} P \log |Re^{i\varphi} - a_\mu| \, d\varphi - \sum_{\mu=1}^{m} p_\mu \log \left| \frac{R^2 - \bar{a}_\mu re^{i\theta}}{R(re^{i\theta} - a_\mu)} \right|$$

$$+ \sum_{\nu=1}^{n} q_\nu \frac{1}{2\pi} \int_0^{2\pi} P \log \left| \frac{1}{Re^{i\varphi} - b_\nu} \right| d\varphi + \sum_{\nu=1}^{n} q_\nu \log \left| \frac{R^2 - \bar{b}_\nu re^{i\theta}}{R(re^{i\theta} - b_\nu)} \right|$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} P \log |T(Re^{i\varphi})| \, d\varphi \; [\text{by cases I, II, III}]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} P \log \left| \frac{(Re^{i\varphi} - a_1)^{p_1} \cdots (Re^{i\varphi} - a_m)^{p_m}}{(Re^{i\varphi} - b_1)^{q_1} \cdots (Re^{i\varphi} - b_n)^{q_n}} T(Re^{i\varphi}) \right| d\varphi$$

$$- \sum_{\mu=1}^{m} p_\mu \log \left| \frac{R^2 - \bar{a}_\mu re^{i\theta}}{R(re^{i\theta} - a_\mu)} \right| + \sum_{\nu=1}^{n} q_\nu \log \left| \frac{R^2 - \bar{b}_\nu re^{i\theta}}{R(re^{i\theta} - b_\nu)} \right|$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\theta - \varphi)} \log |f(Re^{i\varphi})| \, d\varphi$$

$$- \sum_{\mu=1}^{m} p_\mu \log \left| \frac{R^2 - \bar{a}_\mu re^{i\theta}}{R(re^{i\theta} - a_\mu)} \right| + \sum_{\nu=1}^{n} q_\nu \log \left| \frac{R^2 - \bar{b}_\nu re^{i\theta}}{R(re^{i\theta} - b_\nu)} \right|$$

This proves the Theorem.

Note This formula is deduced under the assumption that neither the poles b_j nor the zeros a_μ are to be found on $|z|=R$. The Theorem continues to hold if we permit zeros and poles on $|z|=R$. From now on we shall make these interpretation.

Further if there are multiple zeros and poles in $|z|\le R$, the result will not be affected provided we agree to count each multiple zero and multiple pole ~~as~~ a number of times, equal to its order.

Jensen's Theorem

Theorem Let f be analytic in $|z|<R_1$ and $f(0)\neq 0$. Let $r_1, r_2, \dots, r_n, \dots$ be the modulii of the zeros of f in the disc $|z|<R_1$, arranged in the nondecreasing order. If ~~$r_m \le R < r_{m+1}$~~ $r_m \le R < r_{m+1}$,

$$\log \frac{R^m |f(0)|}{r_1 r_2 \cdots r_m} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})|\,d\varphi,$$

where a zero of order p is counted p times.

Proof Since f has no pole, putting $r=0$ in P.J. formula we get

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})|\,d\varphi - \sum_{\mu=1}^{m} \log\frac{R}{|a_\mu|}$$

i.e. $$\log \frac{|f(0)|\,R^m}{r_1 r_2 \cdots r_m} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})|\,d\varphi$$

Alternative statement

Let $n(x)$ denote the number of zeros of f in $|z|\le x$. $r_m \le R < r_{m+1}$

$$\int_0^R \frac{n(x)}{x}\,dx = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)| \;;\; f(0)\neq 0.$$

Proof Evidently $n(x)$ is a nondecreasing fn of x which is constant in any interval that does not contain the ~~modulii~~ modulii of zeros of f. So if $r_m \le R \le r_{m+1}$

$$\int_0^R \frac{n(x)}{x}\,dx = \lim_{\varepsilon\to 0}\left[\int_0^{r_1-\varepsilon} \frac{n(x)}{x}\,dx + \int_{r_1}^{r_2-\varepsilon} \frac{n(x)}{x}\,dx + \cdots + \int_{r_{m-1}}^{r_m-\varepsilon} \frac{n(x)}{x}\,dx + \int_{r_m}^{R-\varepsilon} dx\right]$$

$$= \lim_{\varepsilon\to 0}\left[\int_0^{r_1-\varepsilon} \frac{0}{x}\,dx + \int_{r_1}^{r_2-\varepsilon} \frac{1}{x}\,dx + \cdots + \int_{r_{m-1}}^{r_m-\varepsilon} \frac{m-1}{x}\,dx + \int_{r_m}^{R-\varepsilon} \frac{m}{x}\,dx\right]$$

$$= \lim_{\varepsilon\to 0}\left[\log(r_2-\varepsilon) - \log r_1 + 2\log(r_3-\varepsilon) - 2\log r_2 + \cdots + (m-1)\log(r_m-\varepsilon) - (m-1)\log r_{m-1} + m\log(R-\varepsilon) - m\log r_m\right]$$

$$= m\log R - \log r_1 - \log r_2 - \log r_3 - \cdots - \log r_m = \log\frac{R^m}{r_1 r_2 \cdots r_m}$$

So from Jensen's theorem we get,

$$\log\frac{R^m}{r_1 r_2 \cdots r_m} + \log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})|\,d\varphi$$

i.e., $\displaystyle\int_0^R \frac{n(x)}{x}\,dx = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$

<u>Alternative proof.</u>

Evidently $n(x)$ is a nondecreasing fn of x which is constant in any interval that does not contain the moduli of zeros of f.

We have $n(x)=0$ for $x<r_1$. If $r_n < r_{n+1}$ then $n(x) = n$ in $r_n \le x < r_{n+1}$. Let $r_m \le R < r_{m+1}$.

Let $r_1 = \alpha_1, \alpha_2, \ldots, \alpha_N = r_m$ be the distinct members of the set $E = \{r_1, r_2, \ldots, r_m\}$. Suppose α_i is repeated p_i times in the set E. Then $p_1 + p_2 + \cdots + p_N = m$. Let $s_i = p_1 + p_2 + \cdots + p_i$; $i = 1, 2, \ldots, N$ $(s_N = m)$.

Let $r_m < R$. Then

$$\int_0^R \frac{n(t)\,dt}{t} = \lim_{\varepsilon\to 0}\Big[\int_{\alpha_1}^{\alpha_2-\varepsilon} \frac{n(t)}{t}dt + \int_{\alpha_2}^{\alpha_3-\varepsilon}\frac{n(t)}{t}dt + \cdots + \int_{\alpha_{N-1}}^{\alpha_N-\varepsilon}\frac{n(t)}{t}dt$$
$$+ \int_{\alpha_N}^{R-\varepsilon} \frac{n(t)}{t}\,dt\Big]$$

$$= \lim_{\varepsilon\to 0}\left[\int_{\alpha_1}^{\alpha_2-\varepsilon}\frac{s_1}{t}dt + \int_{\alpha_2}^{\alpha_3-\varepsilon}\frac{s_2\,dt}{t} + \cdots + \int_{\alpha_{N-1}}^{\alpha_N-\varepsilon}\frac{s_{N-1}}{t}dt + \int_{\alpha_N = r_m}^{R-\varepsilon}\frac{m}{t}dt\right]$$

$$= \lim_{\varepsilon\to 0}\Big\{ s_1 \log t\Big]_{\alpha_1}^{\alpha_2-\varepsilon} + s_2 \log t\Big]_{\alpha_2}^{\alpha_3-\varepsilon} + \cdots + s_{N-1}\log t\Big]_{\alpha_{N-1}}^{\alpha_N-\varepsilon}$$
$$+ m \log t\Big]_{r_m}^{R-\varepsilon}\Big\}$$

$$= \lim_{\varepsilon\to 0}\big[s_1(\log(\alpha_2-\varepsilon) - \log\alpha_1) + s_2(\log(\alpha_3-\varepsilon) - \log\alpha_2) + \cdots$$
$$+ s_{N-1}(\log(\alpha_N-\varepsilon) - \log\alpha_{N-1}) + m(\log(R-\varepsilon) - \log r_m)\big]$$

$$= s_1\log\alpha_2 - s_1\log\alpha_1 + s_2\log\alpha_3 - s_2\log\alpha_2 + \cdots + s_{N-1}\log\alpha_N$$
$$- s_{N-1}\log\alpha_{N-1} + m(\log R - \log r_m).$$

$$= p_1\log\alpha_2 - p_1\log\alpha_1 + (p_1+p_2)\log\alpha_3 - (p_1+p_2)\log\alpha_2 + \cdots$$
$$+ (p_1+p_2+\cdots+p_{N-1})\log\alpha_N - \cdots (p_1+\cdots+p_{N-1})\log\alpha_{N-1}$$
$$+ m\log R - m\log r_m \quad (m = p_1+\cdots+p_N)$$

$$= m\log R - (p_1\log\alpha_1 + p_2\log\alpha_2 + \cdots + p_N\log\alpha_N)$$

$$= \log\frac{R^m}{\alpha_1^{p_1}\alpha_2^{p_2}\cdots\alpha_N^{p_N}} = \log\frac{R^m}{r_1 r_2\cdots r_m}$$

If $R = r_m$, then $\displaystyle\int_0^R \frac{n(t)}{t}dt$

$$= \lim_{\varepsilon\to 0}\left[\int_{\alpha_1}^{\alpha_2-\varepsilon}\frac{s_1\,dt}{t} + \int_{\alpha_2}^{\alpha_3-\varepsilon}\frac{s_2\,dt}{t} + \cdots + \int_{\alpha_{N-1}}^{\alpha_N-\varepsilon}\frac{s_{N-1}}{t}dt\right].$$

$$= \sum_{i=1}^{N-1} s_i(\log\alpha_{i+1} - \log\alpha_i) + s_N(\log R - \log r_m)$$

$$= \log \frac{R^m}{r_1 r_2 \cdots r_m}$$

Thus in any case $\int_0^R \frac{n(t)}{t}\, dt = \log \frac{R^m}{r_1 r_2 \cdots r_m}$

So from Jensen's ~~formula~~ theorem we get

$$\int_0^R \frac{n(x)}{x}\, dx = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\varphi})|\, d\varphi - \log |f(0)|.$$

General Jensen Formula.

Theorem Let f have zeros at points $a_1, a_2, \ldots, a_m$ and poles at $b_1, b_2, \ldots, b_n$ inside the disc $|z| \le R$ and be analytic elsewhere inside and on the boundary of the disc. If $f(0) \ne 0, \infty$ then

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\varphi})|\, d\varphi + \sum_{\mu=1}^{m} \log \frac{|a_\mu|}{R} - \sum_{\nu=1}^{n} \log \frac{|b_\nu|}{R}$$

The Theory of m.f. depends largely on this formula. This formula is obtained from Poisson – Jensen's formula putting $r = 0$.

If there is no pole then the formula reduces to ordinary Jensen's formula.

Ex Let f be regular and $|f(z)| \le M$ in $|z| \le R$ and suppose that $f(0) \ne 0$. Show that the number of zeros in the disc $|z| \le \delta R$ $(0 < \delta < 1)$ does not exceed $\frac{1}{\log 1/\delta} \log \frac{M}{|f(0)|}$.

Soln Let the modulii of zeros of f in $|z| \le R$ be $r_1, r_2, \ldots, r_N$. Then by Jensen's theorem

$$\log \frac{R^N}{r_1 r_2 \cdots r_N} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\varphi})|\, d\varphi - \log |f(0)|$$

$$\le \log M - \log |f(0)| = \log \frac{M}{|f(0)|} \quad \cdots\cdots (1)$$

Let the modulii of zeros of f in $|z| \le \delta R$ $(0 < \delta < 1)$ be $r_1, r_2, \ldots, r_n$. Then

$$\log \frac{R^n}{r_1 r_2 \cdots r_n} \le \log \frac{R^N}{r_1 r_2 \cdots r_N} \quad \left[\because \frac{R}{r_i} \ge 1 \text{ and } n \le N\right]$$

Now $\log \frac{R^n}{r_1 r_2 \cdots r_n} \ge \log \left(\frac{1}{\delta}\right)^n \quad \left[\because \frac{R}{r_i} \ge \frac{R}{\delta R} = \frac{1}{\delta} \text{ for } i = 1, 2, \ldots, n;\ r_i \le \delta R\right]$

Therefore $n \log \frac{1}{\delta} \le \log \frac{R^N}{r_1 r_2 \cdots r_N} \le \log \frac{M}{|f(0)|}$

i.e. $n \le \frac{1}{\log \frac{1}{\delta}} \cdot \log \frac{M}{|f(0)|}$

The Characteristic Function (Nevanlinna's Characteristic Function)

Let f be a meromorphic function and not constant in the complex plane with zeros and poles at the points $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ (other than 0) respectively, arranged in nondecreasing moduli.

Let $b_1, b_2, \ldots, b_n$ be the poles lying in $|z| \le R$. We write $n(t, f)$ for the number of poles of f in $|z| \le t$, a pole of order p being counted p-times, and

$$N(R, f) = \sum_{\nu=1}^{n} \log \frac{R}{|b_\nu|} = \int_0^R \frac{n(t, f)}{t}\, dt .$$

Since the poles of $\frac{1}{f}$ are the zeros of f, if $a_1, a_2, \ldots, a_m$ are the zeros of f in $|z| \le R$ we get

$$N\left(R, \frac{1}{f}\right) = \sum_{\mu=1}^{m} \log \frac{R}{|a_\mu|} = \int_0^R \frac{n(t, \frac{1}{f})}{t}\, dt .$$

We ~~write~~ define $\log^+ x = \log x$ if $x \geq 1$

$= 0$ if $0 \le x < 1$.

The following properties are obvious

1. $\log^+ x \geq 0$ if $x \geq 0$
2. $\log^+ x \geq \log x$ if $x > 0$
3. $\log^+ x \geq \log^+ y$ if $x > y$.
4. $\log x = \log^+ x - \log^+ \frac{1}{x}$ if $x > 0$.

Thus $$\frac{1}{2\pi}\int_0^{2\pi} \log |f(Re^{i\varphi})|\, d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(Re^{i\varphi})|\, d\varphi - \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(Re^{i\varphi})|}\, d\varphi .$$

We write $m(R, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(Re^{i\varphi})|\, d\varphi$.

With these notations general Jensen's formula becomes

$$\log |f(0)| = \left(m(R, f) - m\left(R, \frac{1}{f}\right)\right) + N(R, f) - N\left(R, \frac{1}{f}\right).$$

i.e, $m(R, f) + N(R, f) = m\left(R, \frac{1}{f}\right) + N\left(R, \frac{1}{f}\right) + \log |f(0)|$, $f(0) \neq 0, \infty$.

Now we put $T(R, f) = m(R, f) + N(R, f)$. Then general Jensen's formula becomes simply

$$T(R, f) = T\left(R, \frac{1}{f}\right) + \log |f(0)|, \quad \ldots\ldots (1)$$

where $f(0) \neq 0, \infty$.

The function $T(R, f)$ is called Nevanlinna's characteristic function of f.

Note-1 The formula (1) states that the characteristic functions of f and $\frac{1}{f}$ differ by a constant.

Note-2 If f is an entire function $N(R, f) = 0$ and

$T(R, f) = m(R, f) \le \log^+ M(R, f)$.

Note-3 Since $m(R, f) \geq 0$ $T(R, f) \geq N(R, f) \geq 0$. i.e, the

the function $T(r,f)$ is a non-negative real function.

<u>Theorem</u> If $f_1, f_2, \ldots, f_p$ are meromorphic functions, then

(1) $m\left\{r, \sum_{\nu=1}^{p} f_\nu\right\} \leq \sum_{\nu=1}^{p} m(r, f_\nu) + \log p.$

$m\left\{r, \prod_{\nu=1}^{p} f_\nu\right\} \leq \sum_{\nu=1}^{p} m(r, f_\nu).$

(2) $N\left\{r, \sum_{\nu=1}^{p} f_\nu\right\} \leq \sum_{\nu=1}^{p} N(r, f_\nu)$

$N\left\{r, \prod_{\nu=1}^{p} f_\nu\right\} \leq \sum_{\nu=1}^{p} N(r, f_\nu).$

(3) $T\left\{r, \sum_{\nu=1}^{p} f_\nu\right\} \leq \sum_{\nu=1}^{p} T(r, f_\nu) + \log p$

$T\left\{r, \prod_{\nu=1}^{p} f_\nu\right\} \leq \sum_{\nu=1}^{p} T(r, f_\nu).$

To prove the above result we need the following lemma.

<u>Lemma</u>. If $a_1, a_2, \ldots, a_p$ are any $p(\geq 1)$ complex numbers. then (i) $\log^+ \left|\prod_{\nu=1}^{p} a_\nu\right| \leq \sum_{\nu=1}^{p} \log^+ |a_\nu|$, /

(ii) $\log^+ \left|\sum_{\nu=1}^{p} a_\nu\right| \leq \log^+\left(p \max_{1\leq\nu\leq p} |a_\nu|\right)$

$\leq \sum_{\nu=1}^{p} \log^+ |a_\nu| + \log p.$

<u>Proof</u> If $p=1$, the result trivially holds. So we assume suppose that $p>1$. In this case it is sufficient to prove the result for any two complex numbers.

Let a_1, a_2 be any two complex numbers. We then have to consider the following cases.

<u>Case I</u> $|a_1| \geq 1$, $|a_2| \geq 1$.

Then $\log^+ |a_1 a_2| = \log^+ \{|a_1|\,|a_2|\}$

$= \log \{|a_1|\,|a_2|\} = \log |a_1| + \log |a_2|$

$= \log^+ |a_1| + \log^+ |a_2|.$

<u>Case II</u> $|a_1| \geq 1$, $|a_2| < 1$.

If $|a_1| = 1$ and $|a_2| < 1$ then $\log^+ |a_1 a_2| = \log^+ \{|a_1|\,|a_2|\}$

$= 0 = \log^+ |a_1| + \log^+ |a_2|.$

Next if $|a_1| > 1$, $|a_2| < 1$ then the following possibilities may occur :- (i) $|a_1|\,|a_2| > 1$, (ii) $|a_1| \cdot |a_2| < 1$, (iii) $|a_1| \cdot |a_2| = 1$

(i) If $|a_1| \cdot |a_2| > 1$ then $\log^+ |a_1 a_2| = \log^+ \{|a_1|\,|a_2|\}$

$= \log \{|a_1|\,|a_2|\} = \log |a_1| + \log |a_2| < \log |a_1|$

$= \log^+ |a_1| + \log^+ |a_2|$ $[\because \log^+ |a_2| = 0]$

(ii) If $|a_1|\,|a_2| < 1$ then $\log^+ |a_1 a_2| = \log^+ \{|a_1|\,|a_2|\} = 0$

$< \log^+ |a_1| + \log^+ |a_2|$ $[\because \log^+ |a_1| > 0$ and $\log^+ |a_2| = 0]$.

(iii) If $|a_1||a_2| = 1$, $\log^+|a_1 a_2| = \log^+\{|a_1||a_2|\} = 0$

$< \log^+|a_1| + \log^+|a_2|$ $[\because \log^+|a_1| > 0$ and $\log^+|a_2| = 0]$

Thus in case II we get $\log^+|a_1 a_2| \leq \log^+|a_1| + \log^+|a_2|$.

Case III $|a_1| < 1$, $|a_2| \geq 1$.

The same conclusion holds as in Case II.

Case IV $|a_1| < 1$ and $|a_2| < 1$.

Now, $\log^+|a_1 a_2| = \log^+\{|a_1||a_2|\} = 0 = \log^+|a_1| + \log^+|a_2|$

$[\because \log^+|a_1| = \log^+|a_2| = 0]$.

Considering all these cases together we conclude that $\log^+|a_1 a_2| \leq \log^+|a_1| + \log^+|a_2|$, where a_1, a_2 are any two complex numbers.

This proves the first part of the lemma.

Again $\log^+\left|\sum_{\nu=1}^{p} a_\nu\right| \leq \log^+(|a_1| + |a_2| + \cdots + |a_p|)$

$$\leq \log^+\left(p \max_{\nu=1,2,\ldots,p} |a_\nu|\right)$$

$\leq \log^+ p + \log^+\left(\max_{\nu=1,2,\ldots,p} |a_\nu|\right)$ (by the first part of the lemma)

$\leq \log p + \sum_{\nu=1}^{p} \log^+|a_\nu|$ $\left[\because \log^+ \max_{\nu=1,2,\ldots,p} |a_\nu| \text{ is one of the terms of } \sum_{\nu=1}^{p} \log^+|a_\nu|\right]$

This proves the lemma.

Proof of (1) We get by definition

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\varphi})|\,d\varphi.$$

So, $$m\left(r, \sum_{\nu=1}^{p} f_\nu\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\sum_{\nu=1}^{p} f_\nu(re^{i\varphi})\right| d\varphi$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \left\{\sum_{\nu=1}^{p} \log^+|f_\nu(re^{i\varphi})| + \log p\right\} d\varphi$$

$$= \sum_{\nu=1}^{p} \frac{1}{2\pi}\int_0^{2\pi} \log^+|f_\nu(re^{i\varphi})|\,d\varphi + \frac{1}{2\pi}\int_0^{2\pi} \log p\,d\varphi$$

$$= \sum_{\nu=1}^{p} m(r, f_\nu) + \log p.$$

Also $$m\left(r, \prod_{\nu=1}^{p} f_\nu\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\prod_{\nu=1}^{p} f_\nu(re^{i\varphi})\right| d\varphi$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \sum_{\nu=1}^{p} \log^+|f_\nu(re^{i\varphi})|\,d\varphi \leq \sum_{\nu=1}^{p} \frac{1}{2\pi}\int_0^{2\pi} \log^+|f_\nu(re^{i\varphi})|\,d$$

$$= \sum_{\nu=1}^{p} m(r, f_\nu).$$

Proof of (2) We get by definition

$N(R,f) = \int_0^r \frac{n(t,f)}{t}\,dt$, where $n(t,f)$ is the num of poles of f in $|z| \leq t$.

Clearly if f is the sum or product of the fu f_ν then the order of a pole of f at a point is at most equal to the sum of the orders of the poles of f_ν at z. Thus it follows that

$$n(t, f) \le \sum_{\nu=1}^{p} n(t, f_\nu).$$

So on integration we get

$$N\left\{r, \sum_{\nu=1}^{p} f_\nu\right\} \le \sum_{\nu=1}^{p} N(r, f_\nu) \text{ and}$$

$$N\left\{r, \prod_{\nu=1}^{p} f_\nu\right\} \le \sum_{\nu=1}^{p} N(r, f_\nu).$$

Proof of (3)

We have by definition $T(r, f) = m(r, f) + N(r, f)$.

Therefore,

$$T\left\{r, \sum_{\nu=1}^{p} f_\nu\right\} = m\left\{r, \sum_{\nu=1}^{p} f_\nu\right\} + N\left\{r, \sum_{\nu=1}^{p} f_\nu\right\}$$

$$\le \sum_{\nu=1}^{p} m(r, f_\nu) + \sum_{\nu=1}^{p} N(r, f_\nu), \text{ by (1) and (2).} + \log p$$

$$= \sum_{\nu=1}^{p} T(r, f_\nu) + \log p.$$

Also,

$$T\left\{r, \prod_{\nu=1}^{p} f_\nu\right\} = m\left\{r, \prod_{\nu=1}^{p} f_\nu\right\} + N\left\{r, \prod_{\nu=1}^{p} f_\nu\right\}$$

$$\le \sum_{\nu=1}^{p} m(r, f_\nu) + \sum_{\nu=1}^{p} N(r, f_\nu) = \sum_{\nu=1}^{p} T(r, f_\nu).$$

Some deductions.

In particular taking $p=2$ and $f_1(z) = f(z)$, $f_2(z) = a$, a constant, in (3) we get

$$T(r, f+a) \le T(r, f) + T(r, a) + \log 2$$

$$= T(r, f) + \log^+ |a| + \log 2, \text{ because}$$

$$T(r, a) = m(r, a) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |a| \, d\varphi = \log^+ |a|$$

Since we may replace $f+a$, f by f, $f-a$ and a by $-a$, we get

$$T(r, f) \le T(r, f-a) + \log^+ |a| + \log 2$$

i.e, $T(r, f) - T(r, f-a) \le \log^+ |a| + \log 2$

and $T(r, f-a) \le T(r, f) + \log^+ |a| + \log 2$

i.e, $T(r, f-a) - T(r, f) \le \log^+ |a| + \log 2$.

So we obtain $|T(r, f) - T(r, f-a)| \le \log^+ |a| + \log 2$.

Nevanlinna's first fundamental Theorem.

Theorem Let f be a function meromorphic and not constant in the complex plane, and let a be any complex number. Then

$$m\left(R, \frac{1}{f-a}\right) + N\left(R, \frac{1}{f-a}\right) = T(R, f) - \log |f(0) - a| + \varepsilon(a, R),$$

where $|\varepsilon(a, R)| \le \log^+ |a| + \log 2$.

Proof We get by definition and general Jensen's formula

$$m\left(R, \frac{1}{f-a}\right) + N\left(R, \frac{1}{f-a}\right) = T\left(R, \frac{1}{f-a}\right)$$

$$= T(R, f-a) - \log |f(0) - a|$$

$= T(R, f-a) - T(R,f) + T(R,f) - \log|f(0)-a| \neq$

Example Find $T(r,f)$ of $f(z) = \frac{1}{z-2}$.

Sol^n We know that $m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|re^{i\theta}-2|}\, d\theta$

Since $\frac{1}{|re^{i\theta}-2|} \le \frac{1}{r-2} \to 0$ as $r \to \infty$, we get for all large r, $\log^+ \frac{1}{|re^{i\theta}-2|} = 0$. So $m(r,f) = 0$ for all large r. So in this case

$$N(r,f) = \int_0^r \frac{n(t,f)}{t}\, dt = \int_0^2 \frac{0}{t}\, dt + \int_2^r \frac{1}{t}\, dt = \log\frac{r}{2} \quad (r>2)$$

So, $T(r,f) = m(r,f) + N(r,f) = \log\frac{r}{2}$ for all large r.

$= \log r + O(1)$ for all large r.

Ex If f is a polynomial of degree n, prove that $T(r,f) = O(n\log r)$.

Sol^n Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $a_n \neq 0$.

Since f is an entire function $N(r,f) = 0$. Then

$$T(r,f) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |a_0 + a_1 re^{i\theta} + a_2 r^2 e^{i2\theta} + \cdots + a_n r^n e^{in\theta}|\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |r^n e^{in\theta}\{\frac{a_0}{r^n} e^{-in\theta} + \frac{a_1}{r^{n-1}} e^{-i(n-1)\theta} + \cdots + a_n\}|\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ r^n |\frac{a_0}{r^n} e^{-in\theta} + \frac{a_1}{r^{n-1}} e^{-i(n-1)\theta} + \cdots + a_n|\, d\theta$$

Now $|\frac{a_0}{r^n} e^{-in\theta} + \frac{a_1}{r^{n-1}} e^{-i(n-1)\theta} + \cdots + a_n| < K$ for sufficiently large r and for all θ, because

$$|\frac{a_0}{r^n} e^{-in\theta} + \frac{a_1}{r^{n-1}} e^{-i(n-1)\theta} + \cdots + a_n|$$

$$\le \frac{|a_0|}{r^n} + \frac{|a_1|}{r^{n-1}} + \cdots + \frac{|a_{n-1}|}{r} + |a_n| \to |a_n| \text{ as } r \to$$

From (1) we get for all sufficiently large values of

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ r^n |\frac{a_0}{r^n} e^{-in\theta} + \frac{a_1}{r^{n-1}} e^{-i(n-1)\theta} + \cdots + a_n$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} \log^+ r^n K\, d\theta = \frac{1}{2\pi}\int \log r^n K\, d\theta.$$

$$= n\log r + \log K = n\log r + O(1).$$

i.e, $T(r,f) = O(n\log r)$.

Ex If $f(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are po of degree p and q respectively, prove that $T(r,f) = O(d\log r)$, where $d = \max(p,q)$.

Sol^n Let $f(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_p z^p}{b_0 + b_1 z + b_2 z^2 + \cdots + b_q z^q}$, $a_p \neq 0$, $b_q \neq 0$

Now we consider the following cases.

Case I Let $p > q$.

Since $b_0 + b_1 z + \cdots + b_q z^q$ is a polynomial of degree q, by fundamental theorem of classical algebra, it has exactly q zeros. These zeros are poles of f. For sufficiently large r all these poles of f lie within $|z| = r$. So.

$$N(r,f) = \sum_{\nu=1}^{q} \log \frac{r}{|a_\nu|}, \text{ where } a_\nu\text{'s are poles of } f.$$

$$= \log \frac{r^q}{|a_1||a_2|\cdots|a_q|}$$

$$= q\log r - \log(|a_1||a_2|\cdots|a_q|)$$

$$= q\log r + O(1).$$

For $z = re^{i\theta}$,

$$|f(z)| = \left| \frac{a_0 + a_1 re^{i\theta} + a_2 r^2 e^{i2\theta} + \cdots + a_p r^p e^{ip\theta}}{b_0 + b_1 re^{i\theta} + b_2 r^2 e^{i2\theta} + \cdots + b_q r^q e^{iq\theta}} \right|$$

$$= r^{p-q} \left| \frac{\frac{a_0}{r^p} e^{-ip\theta} + \frac{a_1}{r^{p-1}} e^{-i(p-1)\theta} + \cdots + a_p}{\frac{b_0}{r^q} e^{-iq\theta} + \frac{b_1}{r^{q-1}} e^{-i(q-1)\theta} + \cdots + b_q} \right|$$

$$\le r^{p-q} \frac{\frac{|a_0|}{r^p} + \frac{|a_1|}{r^{p-1}} + \cdots + |a_p|}{\left| |b_q| - \frac{|b_{q-1}|}{r} - \cdots - \frac{|b_0|}{r^q} \right|}$$

$$\le r^{p-q} \frac{|a_p| + 1}{||b_q| - 1|}, \text{ for sufficiently large values}$$

of r because $\frac{|a_0|}{r^p} + \frac{|a_1|}{r^{p-1}} + \cdots + \frac{|a_{p-1}|}{r} \to 0$ and $\frac{|b_0|}{r^q} + \frac{|b_1|}{r^{q-1}} + \cdots + \frac{|b_{q-1}|}{r} \to 0$

$$= Br^{p-q}, \quad \text{where } B = \frac{|a_p| + 1}{||b_q| - 1|}$$

For sufficiently large r

$$T(r,f) = N(r,f) + m(r,f)$$

$$= q\log r + O(1) + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

$$\le q\log r + O(1) + \frac{1}{2\pi}\int_0^{2\pi} \log^+ Br^{p-q} \, d\theta$$

$$= q\log r + O(1) + \frac{1}{2\pi}\int_0^{2\pi} \log Br^{p-q} \, d\theta$$

because $Br^{p-q} \to \infty$ as $r \to \infty$

$$= p\log r + \cancel{B - O(1)} \cdot O(1)$$

[i.e. $T(r,f) - p\log r \le B - O(1)$.

So. $\frac{T(r,f)}{p\log r} \le 1 + \frac{B - O(1)}{p\log r}$. Since $1 + \frac{B - O(1)}{p\log r} \to 1$ as $r \to \infty$, it follows that for all large values of r, $1 + \frac{B - O(1)}{p\log r} \le 2$

Therefore for all large values of r, $\frac{T(r,f)}{p\log r} \le 2$

i.e., $T(r,f) = O(b\log r) = O(p\log r)$]

<u>Case II</u> Let $p = q$.

From Case I we get for $z = re^{i\theta}$ $|f(z)| \leq B$ for all sufficiently large values of r.

So for all large values of r

$$T(r,f) = N(r,f) + m(r,f) = q\log r + O(1) + \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta$$

$$\leq q\log r + O(1) + \log^+ B.$$

[ie., $\dfrac{T(r,f)}{q\log r} \leq 1 + \dfrac{O(1) + \log^+ B}{q\log r}$.

Since $\dfrac{O(1)+\log^+ B}{q\log r} \to 0$ as $r \to \infty$, it follows that for all large values of r

$\dfrac{T(r,f)}{q\log r} \leq 2$ i.e. $T(r,f) = O(q\log r)$] $\times$

<u>Case III</u> Let $p < q$. $\dfrac{1}{|f(z)|} \leq C r^{q-p}$.

In this case calculating as in Case I we get for $z = re^{i\theta}$.

$|f(z)| \leq B r^{p-q} < 1$ for all sufficiently large values of r, because $r^{p-q} \to 0$ as $r \to \infty$.

Then for sufficiently large values of r

$$T(r,f) = N(r,f) + m(r,f)$$

$$= q\log r + O(1) + \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta$$

$$= q\log r + O(1), \text{ since } |f(re^{i\theta})| < 1.$$

<u>Remark</u> Actually we can show that

$T(r,f) = d\log r + O(1)$, where $d = \max(p, q)$.]

<u>Case (III)</u> Let $q > p$. Then calculating in the line of Case I, we see that $\dfrac{1}{|f(z)|} \leq C r^{q-p}$

~~for all large values of r~~ ~~< 1 because $r^{p-q} \to 0$ as~~

Also if $b_1, b_2, \dots, b_p$ are the zeros of f, we get

$$N\left(r, \frac{1}{f}\right) = \sum_{\mu=1}^{p} \log\frac{r}{|b_\mu|} = p\log r + O(1).$$

So by the first fundamental Theorem we get

$$T(r,f) = m\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f}\right) + O(1)$$

$$= p\log r + \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|^{-1}\,d\theta + O(1)$$

$$\leq p\log r + \frac{1}{2\pi}\int_0^{2\pi} \log^+ C r^{q-p}\,d\theta + O(1)$$

$$= p\log r + O(1) \text{ for all large values of } r$$

~~because $|f(re^{i\theta})|^{-1} < 1$~~.

Case (II)' If $p = q$, we see as in Case (II) for $z = re^{i\theta}$, $|f(z)| \le B$ for all large values of r.

So, $m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$

$\le B$ for all large values of r.

i.e., $m(r,f) = O(1)$.

So, $T(r,f) = m(r,f) + N(r,f) = q \log r + O(1)$.]

Combining these cases with case (III) we get

$T(r,f) = d \log r + O(1)$.

Ex Find $T(r,f)$ when $f(z) = e^z$.

Soln Since $f(z) = e^z$ is an integral function $N(r,f) = 0$.

Therefore

$$T(r,f) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |e^{re^{i\theta}}|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |e^{r\cos\theta} \cdot e^{ir\sin\theta}|\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ e^{r\cos\theta}\, d\theta = \frac{1}{2\pi}\int_0^{\pi/2} \log^+ e^{r\cos\theta}\, d\theta + \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} \log^+ e^{r\cos\theta}\, d\theta + \frac{1}{2\pi}\int_{3\pi/2}^{2\pi} \log^+ e^{r\cos\theta}\, d\theta.$$

$$= \frac{1}{2\pi}\int_0^{\pi/2} \log e^{r\cos\theta}\, d\theta + \frac{1}{2\pi}\int_{3\pi/2}^{2\pi} \log e^{r\cos\theta}\, d\theta,$$ for all sufficiently large r, because in 0 to $\pi/2$ and in $3\pi/2$ to 2π $\cos\theta \ge 0$ and so $e^{r\cos\theta} \ge 1$ but in $\pi/2$ to $3\pi/2$ $\cos\theta < 0$ and so $e^{r\cos\theta} \le 1$.

$$= \frac{1}{2\pi}\int_0^{\pi/2} r\cos\theta\, d\theta + \frac{1}{2\pi}\int_{3\pi/2}^{2\pi} r\cos\theta\, d\theta$$

$$= \frac{r}{2\pi}\left\{ [\sin\theta]_0^{\pi/2} + [\sin\theta]_{3\pi/2}^{2\pi} \right\} = \frac{r}{2\pi}[1+1] = \frac{r}{\pi}$$

Therefore $T(r,f) = \frac{r}{\pi}$ for sufficiently large r.

Ex Prove that (i) $T(r, af+b) \le T(r,f) + O(1)$

(ii) $T\left(r, \frac{af+b}{cf+d}\right) = T(r,f) + O(1)$.

Where a, b, c, d are complex number with $ad - bc \ne 0$.

Soln (i) $T(r, af+b) \le T(r, af) + T(r, b) + \log 2$ [

$\le T(r, af) + \log^+ |b| + \log 2$

$\le T(r, f) + \log^+ |a| + \log^+ |b| + \log 2$

$\le T(r,f) + O(1)$.

(ii) $T\left(r, \frac{af+b}{cf+d}\right) = T\left(r, \frac{a}{c} + \frac{bc-ad}{c(cf+d)}\right)$ $(ad - bc \ne 0)$

$= T\left(r, \frac{a}{c} + \frac{(bc-ad)/c^2}{f + \frac{d}{c}}\right)$

$= T\left(r, \frac{a}{c} + \frac{k}{f+k'}\right)$ $\left[\frac{bc-ad}{c^2} = k \text{ and } \frac{d}{c} = k'\right]$

$\leq T\left(r, \frac{f+c'}{k}\right) + O(1)$

$\leq T(r,f) + O(1)$; by (i)

Now let $g = \frac{af+b}{cf+d}$, then

$T(r,g) \leq T(r,f) + O(1)$ - - - - (1)

Clearly $f = \frac{dg-b}{-cg+a}$, $ad \neq bc$.

Thus as in above

$T(r,f) \leq T(r,g) + O(1)$ - - - - (2)

i.e. $T(r,g) \geq T(r,f) + O(1)$ - - - - (2)

From (1) & (2) we have

$T\left(r, \frac{af+b}{cf+d}\right) = T(r,f) + O(1)$.

Theorem If f is regular for $|z| \leq R$ and $M(r,f) = \max_{|z|=r} |f(z)|$, then

$T(r,f) \leq \log^+ M(r,f) < \frac{R+r}{R-r} T(R,f)$ $0 \leq r < R$.

Proof Since f is regular for $|z| \leq R$, $N(r,f) = 0$ for $0 \leq r < R$. Thus $T(r,f) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\varphi})| d\varphi$

$\leq \log^+ M(r,f)$.

Thus the first part of the inequality is proved.

To prove the second inequality we note that result is trivially true for $M(r,f) \leq 1$. Suppose that $M(r,f) > 1$ and choose $z_0 = re^{i\theta}$ so that $|f(z_0)| = M(r,f)$. Since f has no poles in $|z| \leq R$, we have by P.J. formula

$z_0 = re^{i\theta}$

$\log^+ M(r,f) = \log |f(z_0)|$.

$= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2} \log |f(Re^{i\varphi})| d\varphi$

$-\sum_{\mu=1}^{m} \log \left| \frac{R^2 - z_0\bar{a}_\mu}{R(z_0-a_\mu)} \right|$ - - - - - - (1)

Now the function $\varphi(z) = \frac{R(z-a_\mu)}{R^2-\bar{a}_\mu z}$ is regular in $|z| \leq R$ and on $|z| = R$.

$|\varphi(z)| = \left| \frac{Re^{i\varphi} - a_\mu}{R - \bar{a}_\mu e^{i\varphi}} \right|$ [$\because z = Re^{i\varphi}$]

$= \left| \frac{R - a_\mu e^{-i\varphi}}{R - \bar{a}_\mu e^{i\varphi}} \right| = \left|\frac{\bar{\alpha}}{\alpha}\right| = 1$; $\alpha = R - \bar{a}_\mu e^{i\varphi}$.

So by Max. modulus theorem $|\varphi(z_0)| < 1$. Hence term of the form $\log \left| \frac{R(z_0-a_\mu)}{R^2 - z_0\bar{a}_\mu} \right|$ is negative

From (1) we get

$\log^+ M(r,f) = \log |f(z_0)|$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|f(Re^{i\varphi})|}{R^2-2rR\cos(\theta-\varphi)+r^2}\,d\varphi + \sum_{\mu=1}^{m} \log\left|\frac{R(z-a_\mu)}{R^2-\bar{a}_\mu z}\right|$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|f(Re^{i\varphi})|}{R^2-2rR\cos(\theta-\varphi)+r^2}\,d\varphi \quad \text{-----}(2)$$

Further $R^2-2rR\cos(\theta-\varphi)+r^2 \ge R^2-2rR+r^2 = (R-r)^2$

Then from (2) we get.

$$\log^+ M(r,f) \le \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{(R-r)^2}\log|f(Re^{i\varphi})|\,d\varphi$$

$$\le \frac{1}{2\pi}\,\frac{R+r}{R-r}\int_0^{2\pi}\log^+|f(Re^{i\varphi})|\,d\varphi \quad [\because \log x \le \log^+ x]$$

$$= \frac{R+r}{R-r}\, m(R,f) = \frac{R+r}{R-r}\, T(R,f),\ 0 \le r < R.$$

Ex Let f be a transcendental m.f. in the open complex plane. Show that

$$\lim_{r\to\infty} \frac{T(r,f)}{\log r} = \infty$$

Soln We consider the following cases.

Case I Let f have no pole. Then by Cauchy's inequality we get.

$$|f^{(n)}(0)| \le \frac{\lfloor n\, M(r,f)}{r^n} \text{ for } n = 0, 1, 2, \cdots$$

So, $\log M(r,f) \ge n\log r + \log \frac{|f^{(n)}(0)|}{\lfloor n}$

and hence $\liminf\limits_{r\to\infty} \frac{\log M(r,f)}{\log r} \ge n$ for $n = \ldots$

Therefore $\lim\limits_{r\to\infty} \frac{\log^+ M(r,f)}{\log r} \ge \lim\limits_{r\to\infty} \frac{\log M(r,f)}{\log r} = \infty$

Since f is entire, we get

$$3T(2r,f) \ge \log^+ M(r,f)$$

So, $\lim\limits_{r\to\infty} \frac{T(2r,f)}{\log 2r} \ge \frac{1}{3}\lim\limits_{r\to\infty} \frac{\log^+ M(r,f)}{\log r} \cdot \lim\limits_{r\to\infty} \frac{\log r}{\log 2r}$

$$= \infty.$$

Case II Let f have only a finite number of poles. Then we can write.

$$f(z) = \frac{g(z)}{p(z)},$$

Where g is an entire function and p is a poly

Since $g(z) = f(z)\,p(z)$, it follows that

$$T(r,g) \le T(r,f) + T(r,p)$$

$$= T(r,f) + O(\log r).$$

Hence $\lim\limits_{r\to\infty} \frac{T(r,f)}{\log r} + O(1) \ge \lim\limits_{r\to\infty} \frac{T(r,g)}{\log r} = \infty.$

and so $\lim\limits_{r\to\infty} \frac{T(r,f)}{\log r} = \infty.$

Case III Let f have infinitely many poles.

Then $T(r,f) \geq N(r,f) \geq N(r,f) - N(\sqrt{r}, f)$

$$= \int_0^r \frac{n(t,f)}{t}\,dt - \int_0^{\sqrt{r}} \frac{n(t,f)}{t}\,dt.$$

$$= \int_{\sqrt{r}}^r \frac{n(t,f)}{t}\,dt \geq \frac{1}{2}\, n(\sqrt{r}, f) \log r.$$

[$\because n(t,f)$ is nondecreasing and $t \in [\sqrt{r}, r]$
$n(t,f) \geq n(\sqrt{r}, f)$]

Therefore $\lim\limits_{r\to\infty} \dfrac{T(r,f)}{\log r} = \infty$ because $n(\sqrt{r}, f) \to \infty$ as $r \to \infty$.

Cartan's Identity

If f is meromorphic in $|z|<R$ then

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta})\,d\theta + \log^+|f(0)|, \quad 0<r<R.$$

This expression of $T(r,f)$ by an integral is known as Cartan's identity.

Proof To prove the theorem we require the following lemma:

Lemma For any a, $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \log|a - e^{i\theta}|\,d\theta = \begin{cases} \log|a| & \text{if } |a|\geq 1 \\ 0 & \text{if } |a|<1 \end{cases}$

$= \log^+|a|$

Proof of the Lemma.

Jensen's theorem applied to $f(z) = a - z$ and $r=1$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|a - e^{i\theta}|\,d\theta = \log|a| \quad \text{if } |a|\geq 1$$
$$= \log|a| - \log|a| = 0 \quad \text{if } |a|<1.$$

Proof of the Theorem

We apply general Jensen's formula to $f(z) - e^{i\theta}$ (θ is real) and obtain

$$\log|f(0) - e^{i\theta}| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi}) - e^{i\theta}|\,d\varphi - N\left(r, \frac{1}{f - e^{i\theta}}\right) + N(r, f - e^{i\theta})$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi}) - e^{i\theta}|\,d\varphi + N(r,\infty) - N(r, e^{i\theta}) \longrightarrow (1)$$

if $f(0) \neq e^{i\theta}$.

Now we divide by 2π and integrate both sides (1) w.r.t. θ. In the resulting repeated integral on the r.h.s. we can interchange the order of integrations. (This is permissible since the ~~double~~ is ~~also~~ integrand is a continuous function of θ, φ) and obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(0) - e^{i\theta}|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi}) - e^{i\theta}|\,d\theta + N(r,\infty) - \frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta})\,d\theta$$

So by the lemma we get

$$\log^+|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\varphi})|\,d\varphi + N(r,\infty) - \frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta})\,d\theta$$

~~So by the lemma we get~~

$$\cancel{\log^+|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}}$$

$$= m(r,\infty) + N(r,\infty) - \frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta})\,d\theta$$

$$= m(r,f) + N(r, f-e^{i\theta}) - \frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta})\,d\theta$$

$$= m(r,f) + N(r,f) - \frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta})\,d\theta$$

$$= T(r,f) - \frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta})\,d\theta, \text{ because } N(r,f-e^{i\theta}) = N(r,f).$$

ie, $$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta})\,d\theta + \log^+|f(0)|$$

This proves the Theorem.

<u>Corollary</u> $\frac{1}{2\pi}\int_0^{2\pi} m(r,e^{i\theta})\,d\theta \le \log 2$.

<u>Proof</u> By Nevanlinna's first fundamental Theorem

$$T(r,f) = m\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f-a}\right) + \log|f(0)-a| - \varepsilon(a,r)$$

where $|\varepsilon(a,r)| \le \log^+|a| + \log 2$.

Putting $a = e^{i\theta}$ and $-\varepsilon(a,r) = G(\theta)$ we get

$$T(r,f) = m(r,e^{i\theta}) + N(r,e^{i\theta}) + \log|f(0) - e^{i\theta}| + G(\theta).$$

Integrating both sides w.r.t. θ and using Cartan's identity and the lemma we get

$$\frac{1}{2\pi}\int_0^{2\pi} T(r,f)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} m(r,e^{i\theta})\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta})\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|f(0) - e^{i\theta}|\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} G(\theta)\,d\theta$$

ie, $$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} m(r,e^{i\theta})\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta})\,d\theta + \log^+|f(0)| + \frac{1}{2\pi}\int_0^{2\pi} G(\theta)\,d\theta.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} m(r,e^{i\theta})\,d\theta + T(r,f) + \frac{1}{2\pi}\int_0^{2\pi} G(\theta)\,d\theta$$

ie, $$\frac{1}{2\pi}\int_0^{2\pi} m(r,e^{i\theta})\,d\theta = -\frac{1}{2\pi}\int_0^{2\pi} G(\theta)\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |G(\theta)|\,d\theta \le \log 2.$$

* + If there is no doubt as which function f is referred to, it is convenient to write $m(R,a)$, $N(R,a)$, $n(R,a)$, $T(R)$ instead of $m\left(R, \frac{1}{f-a}\right)$, $N\left(R, \frac{1}{f-a}\right)$, $n\left(R, \frac{1}{f-a}\right)$, $T(R,f)$ if a is finite and $m(R,\infty)$, $N(R,\infty)$, $n(R,\infty)$ instead of $m(R,f)$, $N(R,f)$, $n(R,f)$. So,

$$m(R,a) = m\left(R, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(Re^{i\varphi})-a|}\, d\varphi.$$

$$m(R,\infty) = m(R,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(Re^{i\varphi})|\, d\varphi.$$

$$N(R,a) = N\left(R, \frac{1}{f-a}\right) = \int_0^R \frac{n\left(t, \frac{1}{f-a}\right)}{t}\, dt.$$

$$= \int_0^R \frac{n(t,a)}{t}\, dt.$$

$N(R,\infty) = N(R,f) = \int_0^R \frac{n(t,f)}{t}\, dt$, where $n(t,a)$ denotes the number of roots of the eqn $f(z)=a$ in $|z| \le t$ and $n(t,\infty) = n(t,f)$ denotes the number of poles of f in $|z| \le t$.

It is understood that in this defn a pole (or a-point) of multiplicity p is counted p times.

Alternative form of Nevanlinna's first fundamental theorem

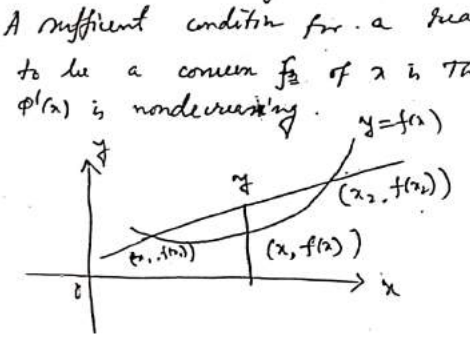
If we allow R to vary the first f. th. can then be written simply as

$$m(R,a) + N(R,a) = T(R) + O(1) \text{ for every}$$

finite or infinite.

Definition A function $y = f(x)$ is called convex downward if for any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the curve $y = f(x)$ the chord joining these two points lies above the arc of the curve between the two points.

Sometimes a function which is convex downward is referred to as convex function. A sufficient condition for a real function $\varphi(x)$ to be a convex fn of x is that $\varphi''(x) > 0$ ie $\varphi'(x)$ is nondecreasing.



Theorem For any a, $N(r,a)$ is an increasing convex f^n of $\log r$.

Proof Now for any a

$$\frac{dN(r,a)}{d\log r} = \frac{dN(r,a)}{dr} \cdot \frac{dr}{d\log r}$$

$$= r\frac{dN(r,a)}{dr}$$

$$= r\frac{d}{dr}\left[\int_0^r \frac{n(t,a)}{t}\,dt\right]$$

$$= r\,\frac{n(r,a)}{r} = n(r,a),$$

which is a nondecreasing function.

So, $N(r,a)$ is a convex function of $\log r$. Also since $\frac{dN(r,a)}{d\log r} = n(r,a) \geq 0$, it follows that $N(r,a)$ is an increasing function of $\log r$.

This proves the Theorem.

Theorem $T(r,f)$ is an increasing convex function of $\log r$.

Proof We see by Cartan's identity that

$$\frac{dT(r,f)}{d\log r} = r\frac{dT(r,f)}{dr} = r\frac{d}{dr}\left[\frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta})\,d\theta + \log^+|f(0)|\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} r\frac{d}{dr}N(r,e^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} n(r,e^{i\theta})\,d\theta = F(r) \text{ say,}$$

Now for $r_1 > r_2$ we get $n(r_1,e^{i\theta}) \geq n(r_2,e^{i\theta})$ for all $\theta \in [0,2\pi]$. So.

$$F(r_1) = \frac{1}{2\pi}\int_0^{2\pi} n(r_1,e^{i\theta})\,d\theta \geq \frac{1}{2\pi}\int_0^{2\pi} n(r_2,e^{i\theta})\,d\theta$$

$$= F(r_2).$$

Hence $F(r)$ and so $\frac{dT(r,f)}{d\log r}$ is a nonnegative and nondecreasing function. Therefore $T(r,f)$ is an increasing convex function of $\log r$. This proves the Theorem.

Note The f^n $m(r,f)$ is not necessarily increasing and convex.

Order of Growth.

Let $S(r)$ be a real valued and nonnegative increasing function for $r_0 < r < \infty$, $r_0 > 0$. The order ρ and the lower order λ of the function $S(r)$ are defined as

$$\rho = \limsup_{r\to\infty} \frac{\log S(r)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r\to\infty} \frac{\log S(r)}{\log r}.$$

The order and the lower order of the function always satisfies the relation $0 \leq \lambda \leq \rho \leq \infty$.

When $\rho = \infty$, $S(r)$ is said to be of infinite order.

If $0 < \rho < \infty$, set $c = \limsup\limits_{r \to \infty} \frac{S(r)}{r^\rho}$ and we distinguish the following possibilities:

(a) $S(r)$ has maximal type of order ρ if $c = +\infty$

(b) $S(r)$ has mean type of order ρ if $0 < c < \infty$

(c) $S(r)$ has minimal type of order ρ if $c = 0$

(d) $S(r)$ has convergence class of order ρ if

$$\int_{r_0}^{\infty} \frac{S(t)}{t^{\rho+1}}\, dt \quad \text{converges.}$$

It follows that if $S(r)$ is of order ρ $(0 < \rho < \infty)$ then for each $\varepsilon (> 0)$

$$S(r) < r^{\rho+\varepsilon} \text{ for all sufficiently large } r$$

and $S(r) > r^{\rho-\varepsilon}$ for a sequence of values of r tending to ∞.

It also follows that if $S(r)$ is of lower order λ $(0 < \lambda < \infty)$ then for each $\varepsilon (> 0)$

$$S(r) > r^{\lambda-\varepsilon} \text{ for all large values of } r$$

and $S(r) < r^{\lambda+\varepsilon}$ for a sequence of values of r tending to ∞.

<u>Theorem</u> If $S(r)$ is of order ρ $(0 < \rho < \infty)$ and has convergence class then $S(r)$ has minimal type of order ρ.

<u>Proof</u> Since $S(r)$ is of convergence class, given $\varepsilon (> 0)$ there exists a number $R = R(\varepsilon)$ such that

$$\int_r^{\infty} \frac{S(t)}{t^{\rho+1}}\, dt < \varepsilon \quad \text{whenever } r > R$$

Then $$\int_r^{2r} \frac{S(t)}{t^{\rho+1}}\, dt < \varepsilon \quad \text{for } r > R$$

Since $S(r)$ is an increasing function, we get

$$\frac{S(r)}{(2r)^{\rho+1}} (2r - r) \le \int_r^{2r} \frac{S(t)}{t^{\rho+1}}\, dt$$

and so $$\frac{S(r)}{r^\rho} < 2^{\rho+1} \cdot \varepsilon \quad \text{for } r > R$$

This implies that $\limsup\limits_{r \to \infty} \frac{S(r)}{r^\rho} = 0$ i.e., $S(r)$ has minimal type. This proves the theorem.

<u>Remark</u> The converse of this theorem is not true. This means that a function of minimal type not always belong to the convergence class.

To clarify the situation let $S(r) = r^\lambda (\log r)^\mu$.

Then order of $S(r)$

$$\rho = \limsup_{r \to \infty} \frac{\log S(r)}{\log r} = \limsup_{r \to \infty} \frac{\lambda \log r + \mu \log(\log r)}{\log r} = \lambda.$$

Also $c = \limsup\limits_{r\to\infty} \dfrac{S(r)}{r^{\lambda}}$

$$= \limsup_{r\to\infty} \frac{r^{\lambda}(\log r)^{\mu}}{r^{\lambda}}$$

$$= \lim_{r\to\infty} (\log r)^{\mu} = \begin{cases} \infty & \text{if } \mu > 0 \\ 1 & \text{if } \mu = 0 \\ 0 & \text{if } \mu < 0 \end{cases}$$

So $S(r)$ is of maximal type if $\mu > 0$

$S(r)$ is of mean type if $\mu = 0$

and $S(r)$ is of minimal type if $\mu < 0$.

Let $-1 < \mu < 0$. Then for $r_0 > 0$

$$\int_{r_0}^{\infty} \frac{S(t)}{t^{\lambda+1}}\,dt = \lim_{R\to\infty} \int_{r_0}^{R} \frac{t^{\lambda}(\log t)^{\mu}}{t^{\lambda+1}}\,dt$$

$$= \lim_{R\to\infty} \int_{r_0}^{R} \frac{(\log t)^{\mu}}{t}\,dt.$$

$$= \lim_{R\to\infty} \int_{\log r_0}^{\log R} x^{\mu}, \quad \text{putting } x = \log t$$

$$= \lim_{R\to\infty} \left[\frac{x^{\mu+1}}{\mu+1}\right]_{\log r_0}^{\log R}$$

$$= \lim_{R\to\infty} \frac{(\log R)^{\mu+1} - (\log r_0)^{\mu+1}}{\mu+1} = \infty \quad \text{because } \mu+1 >$$

So for $-1 < \mu < 0$ Though $S(r)$ is of minimal typ[e] is not of convergence class.

<u>Theorem</u> Let $S(r)$ be of order k $(0 < k < \infty)$. If $k' > k$ then $\displaystyle\int_{r_0}^{\infty} \frac{S(r)}{r^{k'+1}}\,dr$ converges and if $k' < k$ then $\displaystyle\int_{r_0}^{\infty} \frac{S(r)}{r^{k'+1}}\,dr$ diverges.

<u>Proof</u> If $S(r)$ is of order k $(0 < k < \infty)$ then given $\varepsilon\,(>0)$

(i) $S(r) < r^{k+\varepsilon}$ for all large r

(ii) $S(r) > r^{k-\varepsilon}$ for a sequence of values of r

Suppose first that $0 < k < k'$. We put $k' = k + 2\varepsilon$. Then from (i) we get

$$\frac{S(r)}{r^{k'+1}} < \frac{1}{r^{1+\varepsilon}} \text{ for all large values of } r.$$

Since $\varepsilon\,(>0)$, it follows that $\displaystyle\int_{r_0}^{\infty} \frac{S(r)}{r^{k'+1}}\,dr$ converg[es] if $k' > k$ because $\displaystyle\int_{r_0}^{\infty} \frac{dr}{r^{1+\varepsilon}} < \infty$.

For the second part first we suppose that $0 < k' < k < \infty$. Now from (ii) we get $S(r) > r^{k-\varepsilon}$ f a sequence of values of $r \to \infty$.

Letting $k' = k - \varepsilon$ we obtain $S(r) > r^{k-\varepsilon}$ i.e.,

$\dfrac{S(r)}{r^{k'+1}} > \dfrac{1}{r}$ for a sequence of values of $r \to \infty$.

Under this condition the integral $\int_{r_0}^{\infty} \dfrac{S(r)}{r^{k'+1}}\,dr$ diverges. For, if it were convergent then $\lim_{r\to\infty} \dfrac{S(r)}{r^{k'}} = 0$ $(0 < k' < \infty)$ and so for all large r, $S(r) < r^{k'}$ i.e., $\dfrac{S(r)}{r^{k'+1}} < \dfrac{1}{r}$ which contradicts the fact that $\dfrac{S(r)}{r^{k'+1}} > \dfrac{1}{r}$ for a sequence of values of r tending to infinity. So it follows that $\int_{r_0}^{\infty} \dfrac{S(r)}{r^{k'+1}}\,dr$ diverges when $k' < k$, $0 < k' < \infty$.

Next let $k' \le 0$ $(k' \neq -\infty)$. Since $S(r) \to \infty$ as $r \to \infty$, we have for all large r, $S(r) \ge 1$ and so $\dfrac{S(r)}{r} \geq \dfrac{1}{r}$ for all large r.

i.e., $\dfrac{S(r)}{r^{k'+1}} \geq \dfrac{S(r)}{r} \geq \dfrac{1}{r}$ $(\because k' \le 0)$ for all large r.

Hence, $\int_{r_0}^{\infty} \dfrac{S(r)}{r^{k'+1}}\,dr$ diverges when $k' \le 0$ because $\int_{r_0}^{\infty} \dfrac{dr}{r} = \infty$.

Finally let $k' = -\infty$. Then $\dfrac{S(r)}{r^{k'+1}} > \dfrac{1}{r}$ for all [illegible] and hence $\int_{r_0}^{\infty} \dfrac{S(r)}{r^{k'+1}}\,dr$ diverges in this cas[e] also. This proves the theorem.

<u>Theorem</u> If f is an entire function, then the fu[nctions] $S_1(r) = \log^+ M(r,f)$ and $S_2(r) = T(r,f)$ have the same order. Further if the value of the common order be k $(0 < k < \infty)$, $S_1(r)$ and $S_2(r)$ are together minimal type, mean type, maximal type or convergence class.

<u>Proof</u> Let the orders of $S_1(r)$ and $S_2(r)$ be k_1, k_2 respectively. We shall show that $k_1 = k_2$.

Setting $R = 2r$ in $T(r,f) \le \log^+ M(r,f) \le \dfrac{R+r}{R-r}\, T(R,f)$, $0 \le r < R$

we get

$$T(r,f) \le \log^+ M(r,f) \le 3\,T(2r,f)$$

$$\text{i.e., } S_2(r) \le S_1(r) \le 3\,S_2(2r) \quad \cdots\cdots (1)$$

First we suppose that both k_1 & k_2 are finite. From the first inequality of (1) we get

$$\limsup_{r\to\infty} \frac{\log S_2(r)}{\log r} \le \limsup_{r\to\infty} \frac{\log S_1(r)}{\log r} \quad \cdots$$

$$\text{i.e., } k_2 \le k_1 \quad \cdots\cdots (2)$$

Again from the second inequality of (1) we get

$$\limsup_{r\to\infty} \frac{\log S_1(r)}{\log r} \le \limsup_{r\to\infty} \frac{\log 3 + \log S_2(2r)}{\log r}$$

i.e. $$k_1 \le \limsup_{r\to\infty} \frac{\log S_2(2r)}{\log 2r - \log 2}$$

$$= \limsup_{r\to\infty} \frac{\log S_2(2r)}{\log 2r} \cdot \lim_{r\to\infty} \frac{1}{1 - \frac{\log 2}{\log 2r}}$$

$$= k_2 \quad \text{------} (3)$$

From (2) and (3) we therefore have $k_1 = k_2$ when k_1 and k_2 are both finite.

We next show that if the order of one of the function is infinite so is the other.

We get from (1) $S_2(r) \le S_1(r)$ so that if $k_2 = +\infty$ then $k_1 = +\infty$.

Again from (1) we obtain

$$\frac{\log S_1(r)}{\log r} \le \frac{\log 3 + \log S_2(2r)}{\log r}$$

$$= O(1) + \frac{\log S_2(2r)}{\log 2r}\left(1 + \frac{\log 2}{\log r}\right)$$

i.e. $$\limsup_{r\to\infty} \frac{\log S_1(r)}{\log r} \le \limsup_{r\to\infty} \frac{\log S_2(2r)}{\log 2r}$$

and this shows that if $k_1 = +\infty$ then $k_2 = +\infty$.

This proves the first part of the theorem.

Suppose for instance that $S_2(r)$ has convergence class of order k. We now show that $S_1(r)$ has also convergence class of order k. We get from (1)

$$\int_{r_0}^{\infty} \frac{S_1(r)}{r^{k+1}}\,dr \le 3\int_{r_0}^{\infty} \frac{S_2(2r)}{r^{k+1}}\,dr$$

$$= 3\cdot 2^k \int_{2r_0}^{\infty} \frac{S_2(t)}{t^{k+1}}\,dt \quad [\text{putting } t = 2r]$$

$$\le 3\cdot 2^k \int_{r_0}^{\infty} \frac{S_2(t)}{t^{k+1}}\,dt < \infty \quad (0 < k < \infty).$$

Therefore $S_1(r)$ has convergence class of order k.

Conversely if $S_1(r)$ has a convergence class of order k so has $S_2(r)$ because $S_2(r) \le S_1(r)$.

Let $S_1(r)$ be of minimal type. Then

$$\limsup_{r\to\infty} \frac{S_1(r)}{r^k} = 0 \quad (0 < k < \infty). \text{ Hence}$$

$$\limsup_{r\to\infty} \frac{S_2(r)}{r^k} \le \limsup_{r\to\infty} \frac{S_1(r)}{r^k} = 0 \quad [\because S_2(r) \le S_1(r)]$$

So $S_2(r)$ is of minimal type.

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Conversely let $S_1(r)$ be of minimal type

Then $\limsup_{r\to\infty} \frac{S_1(r)}{r^K} = 0$. Now from (1) we get

$$\frac{S_1(r)}{r^K} \leq \frac{3 S_2(2r)}{(2r)^K} \cdot 2^K \text{ so that}$$

$$\limsup_{r\to\infty} \frac{S(r)}{r^K} \leq 3.2^K \limsup_{r\to\infty} \frac{S_2(2r)}{(2r)^K} = 0.$$

So $S_1(r)$ is of minimal type.

Let $S_1(r)$ be of maximal type of order K.

Then $\limsup_{r\to\infty} \frac{S_1(r)}{r^K} = \infty$. Now from (1) we get

$$\frac{S_1(r)}{r^K} \leq 3.2^K \frac{S_2(2r)}{(2r)^K}$$

i.e, $\frac{S_2(2r)}{(2r)^K} \geq \frac{1}{3.2^K} \cdot \frac{S_1(r)}{r^K}$.

i.e, $\limsup_{r\to\infty} \frac{S_2(2r)}{(2r)^K} \geq \frac{1}{3.2^K} \limsup_{r\to\infty} \frac{S_1(r)}{r^K} = \infty$

So. $S_2(r)$ is of maximal type.

Let $S_2(r)$ be of maximal type. Then

$\limsup_{r\to\infty} \frac{S_2(r)}{r^K} = \infty$. Since $S_2(r) \leq S_1(r)$, it follows

that $\infty = \limsup_{r\to\infty} \frac{S_2(r)}{r^K} \leq \limsup_{r\to\infty} \frac{S_1(r)}{r^K}$.

So $S_1(r)$ is of maximal type.

Finally we suppose that $S_1(r)$ is of mean type.

Then $\limsup_{r\to\infty} \frac{S_1(r)}{r^K} = l$, say, $0 < l < \infty$.

Then $\limsup_{r\to\infty} \frac{S_2(r)}{r^K} \leq \limsup_{r\to\infty} \frac{S_1(r)}{r^K} = l < \infty$

Again $0 < l = \limsup_{r\to\infty} \frac{S_1(r)}{r^K} \leq \limsup_{r\to\infty} \frac{3 S_2(2r)}{r^K}$

$$= \limsup_{r\to\infty} \frac{3 S_2(2r)}{(2r)^K} \cdot 2^K$$

i.e, $\limsup_{r\to\infty} \frac{S_2(r)}{r^K} \geq \frac{l}{3.2^K} > 0$

Hence $0 < \limsup_{r\to\infty} \frac{S_2(r)}{r^K} < \infty$ and so $S_2(r)$ is of

mean type.

Next suppose that $S_2(r)$ is of mean type. Then

$\limsup_{r\to\infty} \frac{S_2(r)}{r^K} = m$, $0 < m < \infty$. So.

$$\limsup_{r\to\infty} \frac{S_1(r)}{r^K} \geq \limsup_{r\to\infty} \frac{S_2(r)}{r^K} = m > 0$$

and $\limsup_{r\to\infty} \frac{S_1(r)}{r^K} \leq \limsup_{r\to\infty} \frac{3 S_2(2r)}{(2r)^K} \cdot 2^K = 3.2^K \cdot m < \infty$

Hence $0 < \limsup_{r\to\infty} \frac{S_1(r)}{r^K} < \infty$ and hence

$S_1(r)$ is of mean type.

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Order of a meromorphic function

Let f be a nonconstant meromorphic function in the open complex plane. The function f is said to have order ρ, maximal, mean or minimal type or convergence class if the characteristic function $T(r,f)$ has this property.

It may be noted that for integral function this coincide by the preceding theorem with the corresponding definitions in terms of $\log^+ M(r,f)$.

Accordingly a meromorphic function f is said to be of order ρ if $\limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \rho$.

Accordingly a meromorphic function f is said to be of order ρ if $\limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \rho$.

This implies for $\varepsilon(>0)$ that

$T(r,f) < r^{\rho+\varepsilon}$ for all large r.

and $T(r,f) > r^{\rho-\varepsilon}$ for a sequence of values of r tending to infinity.

Theorem If ρ_1, ρ_2 be the orders of the meromorphic functions $f_1(z)$ and $f_2(z)$ respectively, then

(i) order of $f_1 \pm f_2 \le \max(\rho_1, \rho_2)$.

(ii) order of $f_1 f_2 \le \max(\rho_1, \rho_2)$.

The same also holds for the quotient f_1/f_2.

Proof Since f_1 and f_2 are of orders ρ_1 and ρ_2 respectively we get for $\varepsilon(>0)$ and for all large values of r

$T(r, f_1) < r^{\rho_1+\varepsilon}$ and $T(r, f_2) < r^{\rho_2+\varepsilon}$.

Let $\rho = \max(\rho_1, \rho_2)$. Then for all large values of r we get

$$T(r, f_1 \pm f_2) \le T(r,f_1) + T(r,f_2) + O(1)$$
$$< r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon} + O(1)$$
$$\le r^{\rho+\varepsilon} + r^{\rho+\varepsilon} + r^{\rho+\varepsilon} = 3r^{\rho+\varepsilon}.$$

So $\limsup_{r\to\infty} \frac{\log T(r, f_1 \pm f_2)}{\log r} \le \rho+\varepsilon$. Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log T(r, f_1 \pm f_2)}{\log r} \le \rho.$$

Hence the order of $f_1 \pm f_2 \le \rho = \max(\rho_1, \rho_2)$.

Again $T(r, f_1 f_2) \le T(r, f_1) + T(r, f_2)$
$< r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon} \le 2r^{\rho+\varepsilon}$.

So $\limsup_{r\to\infty} \frac{\log T(r, f_1 f_2)}{\log r} \le \rho+\varepsilon$. Since $\varepsilon(>0)$ is arbitrary, it follows that

$\limsup_{r\to\infty} \frac{\log T(r, f_1 f_2)}{\log r} \leq \rho$. Hence the order of $f_1 f_2 \leq \rho = \max(\rho_1, \rho_2)$.

Also $T(r, f_1/f_2) \leq T(r, f_1) + T(r, 1/f_2)$
$= T(r, f_1) + T(r, f_2) + O(1)$,

by the first fundamental Theorem. So in a similar way we can prove that the order of $f_1/f_2 \leq \rho = \max(\rho_1, \rho_2)$. This proves the Theorem.

<u>Note</u> Let ρ be the order of $f = f_1 f_2$ and $\rho_1 > \rho_2$. Then $\rho \leq \rho_1$, while on the other hand $f_1 = f/f_2$ so that $\rho_1 \leq \max\{\rho, \rho_2\}$. Hence $\rho = \rho_1$. Thus the order of the product of two meromorphic functions f_1 and f_2 of orders ρ_1, ρ_2 respectively is less than or equal to $\max(\rho_1, \rho_2)$ and equality holds provided $\rho_1 \neq \rho_2$.

Similar is the situation for other cases.

<u>Example</u> Determine the order of $\frac{1}{z^2-1}$.

<u>Sol^n</u> In this case $T(r, f) = 2\log r + O(1)$ for all large values of r. Let ρ be the order of f. Then $\rho = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$

$= \limsup_{r\to\infty} \frac{\log(2\log r + O(1))}{\log r} = 0$.

<u>Note</u> In general if f is a rational function then $T(r,f) = O(\log r)$. So the order ρ is

$$\rho = \limsup_{r\to\infty} \frac{\log(O(\log r))}{\log r} = 0.$$

Thus the order of a rational function is zero. In particular, the order of a polynomial is zero.

<u>Theorem</u> Let f be of order ρ and $\varepsilon(>0)$. Then for every a (i) $m(r,a) = O(r^{\rho+\varepsilon})$, (ii) $N(r,a) = O(r^{\rho+\varepsilon})$, (iii) $n(r,a) = O(r^{\rho+\varepsilon})$ and (iv) the series $\sum_n \left(\frac{1}{r_n(a)}\right)^{\rho+\varepsilon}$ is convergent, where $r_1(a), r_2(a), \ldots, r_n(a), \ldots$ are the modulii of the zeros of $f(z)-a$, ordered by increasing magnitude.

<u>Proof</u> Since f is of order ρ we have by def. $T(r,f) = O(r^{\rho+\varepsilon})$ for every $\varepsilon(>0)$ but not for $\varepsilon(<0)$, and for all sufficiently large r.

From the first fundamental Theorem we get $m(r,a) + N(r,a) = T(r,f) + O(1)$. Since

$m(r,a) \le T(r,f) + O(1)$ and $N(r,a) \le T(r,f) + O(1)$ for every a, the results (i) and (ii) follow immediately.

From (ii) we get $N(r,a) < A r^{\rho+\varepsilon}$ ($\varepsilon > 0$) for all sufficiently large r and A is a constant.

Since $n(r,a)$ is a nondecreasing function, we get

$$\int_r^{2r} \frac{n(t,a)}{t} dt \le \int_0^{2r} \frac{n(t,a)}{t} dt = N(2r,a)$$ and

$$\int_r^{2r} \frac{n(t,a)}{t} dt \ge n(r,a) \int_r^{2r} \frac{dt}{t} = n(r,a) \cdot \log 2.$$

Hence $n(r,a) \log 2 \le N(2r,a) < A(2r)^{\rho+\varepsilon}$ so that $n(r,a) = O(r^{\rho+\varepsilon})$. This proves (iii).

Now we shall show that $\sum_n \{r_n(a)\}^{-\alpha}$ is convergent if $\alpha > \rho$.

Let β be a number such that $\rho < \beta < \alpha$.

Then by (iii) we get

$n(r,a) < K r^{\beta}$ for all sufficiently large r.

Putting $r = r_n(a)$ we get

$n < K\{r_n(a)\}^{\beta}$ because $n(r_n(a), a) = n$.

So $\{r_n(a)\}^{-\alpha} < K' n^{-\alpha/\beta}$, where $K' = K^{\alpha/\beta}$.

Since $\alpha/\beta > 1$, it follows by comparison test that $\sum_n \{r_n(a)\}^{-\alpha}$ converges if $\alpha > \rho$. This proves the result.

Note-1 The result shows that for every a the order of each of $m(r,a)$, $N(r,a)$, $n(r,a)$ is at most equal to the order of the given meromorphic function.

Ex Show that every polynomial is of order zero.

Soln Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, $a_n \ne 0$. Let $|z| = r > 1$. Then $|f(z)| \le |a_n| r^n + |a_{n-1}| r^{n-1} + |a_{n-2}| r^{n-2} \cdots + |a_1| r + |a_0|$.

$$\le r^n [|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|]$$

$= K r^n$ where $K = |a_n| + |a_{n-1}| + \cdots + |a_0|$.

So $M(r) \le K r^n$ i.e., $\log M(r) \le \log K + n \log r$

$$\text{So } \rho = \overline{\lim_{r\to\infty}} \frac{\log \log M(r)}{\log r} \le \overline{\lim_{r\to\infty}} \frac{\log[\log K + n\log r]}{\log r}$$

$$= \overline{\lim_{r\to\infty}} \frac{\log\left[n\log r\left(\frac{\log K}{n\log r} + 1\right)\right]}{\log r}$$

$$= \overline{\lim_{r\to\infty}} \frac{\log n + \log\log r + \log\left(\frac{\log K}{n\log r} + 1\right)}{\log r} = 0.$$

Ex Find the order of e^{z^n}, where n is a +ve integer.

Sol: We have $M(r) = \max_{|z|=r} |e^{z^n}| = \max_\theta |e^{r^n(\cos n\theta + i\sin n\theta)}|$

$$= \max_\theta |e^{r^n \cos n\theta} \times e^{i r^n \sin n\theta}| = \max_\theta |e^{r^n \cos n\theta}| = e^{r^n}$$

So, $\log M(r) = r^n$. Hence $\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r}$.

$$= \overline{\lim_{r\to\infty}} \frac{n\log r}{\log r} = n.$$

Hence the order of e^{z^n} is n.

Ex Find the order of e^{e^z}.

Sol: We have $M(r) = \max_{|z|=r} |e^{e^z}| = \max_\theta |e^{e^{r(\cos\theta + i\sin\theta)}}|$

$$= \max_\theta |e^{e^{r\cos\theta} \times e^{ir\sin\theta}}| = \max_\theta |e^{e^{r\cos\theta}\{\cos(r\sin\theta) + i\sin(r\sin\theta)\}}|$$

$$= \max_\theta |e^{e^{r\cos\theta}\cos(r\sin\theta) + i e^{r\cos\theta}\sin(r\sin\theta)}|$$

$$= \max_\theta |e^{e^{r\cos\theta}\cos(r\sin\theta)} \times e^{i e^{r\cos\theta}\sin(r\sin\theta)}|.$$

$$= \max_\theta |e^{e^{r\cos\theta}\cos(r\sin\theta)}| = e^{e^r}.$$ So $\log M(r) = e^r$

Hence, $\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} = \overline{\lim_{r\to\infty}} \frac{r}{\log r} = \infty$.

Comparative growth of $T(r)$ and $\log M(r)$.

Let f be an integral function of order k $(0<k<\infty)$.

Let $c_1 = \overline{\lim_{r\to\infty}} \frac{T(r,f)}{r^k}$, $c_2 = \lim_{r\to\infty} \frac{\log^+ M(r,f)}{r^k}$.

It is to be noted that although for an integral function f, c_1, c_2 are together zero, finite or infinite, they need not have the same value. For example if $f(z) = e^z$, we see that $T(r,f) = \frac{r}{\pi}$ and $\log M(r) = r$ and $k=1$ so that $c_1 = \frac{1}{\pi}$ and $c_2 = 1$.

Since $T(r,f) \le \log^+ M(r,f)$, we get $c_1 \le c_2$.

It is known that

$T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(R,f)$ $(0 \le r < R)$, when f is regular in $|z| \le R$.

Setting $R = r(1+\frac{1}{k})$ in the above result we get

$$T(r,f) \le \log^+ M(r,f) \le \frac{r(2+\frac{1}{k})}{r/k} \cdot T\left(r\left(1+\tfrac{1}{k}\right), f\right).$$

Since $c_1 = \overline{\lim_{r\to\infty}} \frac{T(r,f)}{r^k}$; if c_1 is finite for given $\varepsilon (>0)$

$T(r,f) < (c_1+\varepsilon) r^k$ for all sufficiently large r

Then from above we get

$\log^+ M(r,f) < \frac{(2+\frac{1}{k})}{\frac{1}{k}} (c_1+\varepsilon)\left(1+\frac{1}{k}\right)^k r^k$

$\frac{\log^+ M(r,f)}{r^k} < (2k+1)(c_1+\varepsilon)\left(1+\frac{1}{k}\right)^k$, for all large r.

$c_2 = \overline{\lim_{r\to\infty}} \frac{\log^+ M(r,f)}{r^k} \le (2k+1)(c_1+\varepsilon)\left(1+\frac{1}{k}\right)^k$

$c_2 \le c_1 (2k+1)\left(1+\frac{1}{k}\right)^k$ [$\because \varepsilon (>0)$ is arbitrary]

$< e\,(2k+1)\,c_1$ [$\because \left(1+\frac{1}{k}\right)^k < e$]

Example For $f(z) = e^{e^z}$ show that $\lim_{r\to\infty} \frac{T(r,f)}{\log M(r,f)} = 0$

Sol For $z = re^{i\theta}$ we get

$$|f(re^{i\theta})| = \left|e^{e^{r(\cos\theta + i\sin\theta)}}\right|$$
$$= \left|e^{e^{r\cos\theta}\cdot e^{ir\sin\theta}}\right|$$
$$\le e^{|e^{r\cos\theta}\cdot e^{ir\sin\theta}|}$$
$$= e^{e^{r\cos\theta}}$$

and so $\log^+ |f(re^{i\theta})| \le e^{r\cos\theta}$.

Now $f(z)$ being an integral f^n in the plane

$$T(r,f) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta \text{ and}$$

so $T(r,f) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$ and

so

$$T(r,f) \le \frac{1}{2\pi}\int_0^{2\pi} e^{r\cos\theta}\, d\theta = \frac{1}{2\pi}\int_0^{\pi} e^{r\cos\theta}\, d\theta + \frac{1}{2\pi}\int_\pi^{2\pi} e^{r\cos\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{r\cos\theta}\, d\theta$$ [Putting $\psi = \theta - 2\pi$ in the second integral]

$$= \frac{1}{\pi}\int_0^{\pi} e^{r\cos\theta}\, d\theta$$ [$\because$ the integrand is an even f^n]

$$= \frac{e^r}{\pi}\int_0^{\pi} e^{-r(1-\cos\theta)}\, d\theta$$

By Jordan inequality we see that

$$1-\cos\theta = 2\sin^2\frac{\theta}{2} \ge \frac{2\theta^2}{\pi^2} \quad (\because 0 \le \tfrac{\theta}{2} \le \pi/2)$$

[J. inequality For $0 \le \theta \le \pi/2$ $\frac{2\theta}{\pi} \le \sin\theta \le \theta$]

Thus $T(r,f) \le \frac{e^r}{\pi}\int_0^{\pi} e^{-\frac{2r\theta^2}{\pi^2}}\, d\theta$

$$\le \frac{e^r}{\pi}\int_0^{\infty} e^{-\frac{2r\theta^2}{\pi^2}}\, d\theta$$

$$= \frac{e^r}{\pi}\sqrt{\frac{\pi^3}{8r}} \quad \left[\because \int_0^{\infty} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}\right]$$

$$= e^{r}\sqrt{\frac{\pi}{8r}} \quad \cdots\cdots (1)$$

Also $M(r,f) = \max_{|z|=r} |f(z)| = e^{e^r}$

i.e., $\log M(r,f) = e^{r} \quad \cdots\cdots (2)$

So from (1) & (2) we get

$$\lim_{r\to\infty} \frac{T(r,f)}{\log M(r,f)} \le \lim_{r\to\infty} \sqrt{\frac{\pi}{8r}} = 0.$$

The Second Fundamental Theorem of Nevanlinna.

Th Suppose that f is a non-constant m.f. in $|z| \le R$. Let $a_1, a_2, \ldots, a_q$ $(q \ge 2)$ be distinct finite complex numbers, $|a_\mu - a_\nu| \ge \delta$ $(1 \le \mu < \nu \le q)$ for some $\delta (>0)$. Then for $0 < r < R$

$$m(r,\infty) + \sum_{\nu=1}^{q} m(r, a_\nu) \le 2T(r,f) - N_1(r) + S(r,f), \quad \cdots\cdots (1)$$

where $N_1(r) = N(r, \frac{1}{f'}) + 2N(r,f) - N(r, f')$

and $S(r,f) = m(r, \frac{f'}{f'}) + m\{r, \sum_{\nu=1}^{q} \frac{f'}{f-a_\nu}\}$

$$+ q\log^{+}\frac{3q}{\delta} + \log 2 + \log\frac{1}{|f'(0)|},$$

it being assumed that $f(0) \ne 0, \infty$ and $f'(0) \ne 0$.

Proof Suppose that $F(z) = \sum_{\nu=1}^{q} \frac{1}{f(z)-a_\nu}$

Let for some ν $(\nu = 1, 2, \ldots, q)$, if there is any.

$|f(z) - a_\nu| < \frac{\delta}{3q}$ (z is temporarily fixed on $|z| = r$).

Then for $\mu \ne \nu$

$$|f(z) - a_\mu| = |(a_\nu - a_\mu) - (a_\nu - f(z))|$$
$$\ge |a_\mu - a_\nu| - |f(z) - a_\nu|$$
$$> \delta - \frac{\delta}{3q} \quad [\because |f(z) - a_\nu| < \frac{\delta}{3q}]$$
$$\ge \delta - \frac{\delta}{3} \quad [\because q \ge 2]$$
$$= \frac{2\delta}{3} \quad \cdots\cdots (2)$$

Therefore, for $\mu \ne \nu$

$$\frac{1}{|f(z) - a_\mu|} < \frac{3}{2\delta} < \frac{1}{2q}\left\{\frac{1}{|f(z) - a_\nu|}\right\} \cdots (3)$$

$$[\because q\,|f(z) - a_\nu| < \frac{\delta}{3}].$$

Again $|F(z)| = \left|\frac{1}{f(z) - a_\nu} + \sum_{\mu\ne\nu} \frac{1}{f(z) - a_\mu}\right|$

$$\ge \frac{1}{|f(z) - a_\nu|} - \sum_{\mu\ne\nu} \frac{1}{|f(z) - a_\mu|}$$

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$$\geq \frac{1}{|f(z)-a_\nu|} - \frac{q-1}{2q} \cdot \frac{1}{|f(z)-a_\nu|} \quad [\text{by } (3)]$$

$$= \left\{1 - \frac{q-1}{2q}\right\} \frac{1}{|f(z)-a_\nu|}$$

$$\geq \frac{1}{2} \frac{1}{|f(z)-a_\nu|} \quad \left[\because \frac{q+1}{2q} > \frac{1}{2}\right]$$

i.e, $2|F(z)| \geq \frac{1}{|f(z)-a_\nu|}$

i.e, $\log^+(2|F(z)|) \geq \log^+ \frac{1}{|f(z)-a_\nu|}$

i.e, $\log^+ 2 + \log^+|F(z)| \geq \log^+ \frac{1}{|f(z)-a_\nu|}$

i.e. $\log^+ 2 + \log^+|F(z)| \geq \log^+ \frac{1}{|f(z)-a_\nu|}$

i.e, $\log^+|F(z)| \geq \log^+ \frac{1}{|f(z)-a_\nu|} - \log^+ 2$

i.e, $\log^+|F(z)| \geq \sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z)-a_\mu|} - \sum_{\mu\neq\nu} \log^+ \frac{1}{|f(z)-a_\mu|} - \log 2$

Since $|f(z)-a_\mu| \geq \frac{2\delta}{3} > \frac{\delta}{2}$ for $\mu \neq \nu$,

$\log^+ \frac{1}{|f(z)-a_\mu|} \leq \log^+ \frac{2}{\delta}$ for $\mu \neq \nu$.

So. $\log^+|F(z)| \geq \sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z)-a_\mu|} - \log 2 - (q-1)\log^+\frac{2}{\delta}$

$$\geq \sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z)-a_\mu|} - q\log^+\frac{2}{\delta} - \log 2$$

$$\geq \sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z)-a_\mu|} - q\log^+\frac{3q}{\delta} - \log 2 \quad \text{.....} (4)$$

This is true if $|f(z)-a_\nu| < \frac{\delta}{3q}$ for some $\nu \leq q$. This inequality is true evidently for at most one ν. If it is not true for any value of ν then $|f(z)-a_\mu| \geq \frac{\delta}{3q}$ i.e. $\frac{1}{|f(z)-a_\mu|} \leq \frac{3q}{\delta}$ for $\mu=1,2,\dots,q$.

Therefore, $\sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z)-a_\mu|} \leq q \log^+ \frac{3q}{\delta}$ so that

$\sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z)-a_\mu|} - q\log^+ \frac{3q}{\delta} - \log 2 \leq 0$

Therefore,

$\log^+|F(z)| \geq \sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z)-a_\mu|} - q\log^+\frac{3q}{\delta} - \log 2$

$[\ \log^+|F(z)| \geq 0\ ]$

So inequality (4) holds in all cases. Integrating we deduce

$$m(r,F) \geq \sum_{\mu=1}^{q} m(r,a_\mu) - q\log^+\frac{3q}{\delta} - \log 2 \quad \cdots (5)$$

Again $m(r,F) = m\left(r, \frac{1}{f}\cdot\frac{f}{f'}\cdot f'F\right)$

$$\leq m\left(r,\frac{1}{f}\right) + m\left(r,\frac{f}{f'}\right) + m(r, f'F)$$

Also we get by general Jensen's formula

$$T(r,f) = T\left(r,\frac{1}{f}\right) + \log|f(0)|$$

So, $T\left(r,\frac{f}{f'}\right) = T\left(r,\frac{f'}{f}\right) + \log\left|\frac{f(0)}{f'(0)}\right|$

i.e. $m\left(r,\frac{f}{f'}\right) = m\left(r,\frac{f'}{f}\right) + N\left(r,\frac{f'}{f}\right) - N\left(r,\frac{f}{f'}\right)$

$$+\log\left|\frac{f(0)}{f'(0)}\right|$$

and $m\left(r,\frac{1}{f}\right) = T(r,f) - N\left(r,\frac{1}{f}\right) + \log\frac{1}{|f(0)|}$

Hence we get

$$m(r,F) \leq T(r,f) - N\left(r,\frac{1}{f}\right) + \log\frac{1}{|f(0)|} + m\left(r,\frac{f'}{f}\right)$$
$$+N\left(r,\frac{f'}{f}\right) - N\left(r,\frac{f}{f'}\right) + \log\left|\frac{f(0)}{f'(0)}\right| + m(r,f'F) \cdots (6)$$

Combining (5) and (6) we obtain

$$\sum_{\mu=1}^{q} m(r,a_\mu) + m(r,\infty) \leq m(r,F) + m(r,f) + q\log^+\frac{3q}{\delta} + \log 2$$
$$\leq T(r,f) - N\left(r,\frac{1}{f}\right) + \log\frac{1}{|f(0)|} + m\left(r,\frac{f'}{f}\right)$$
$$+N\left(r,\frac{f'}{f}\right) - N\left(r,\frac{f}{f'}\right) + \log\left|\frac{f(0)}{f'(0)}\right|$$
$$+m(r,f'F) + m(r,f) + q\log^+\frac{3q}{\delta} + \log 2$$
$$= T(r,f) - N\left(r,\frac{1}{f}\right) + \log\frac{1}{|f(0)|} + m\left(r,\frac{f'}{f}\right) + N\left(r,\frac{f'}{f}\right)$$
$$-N\left(r,\frac{f}{f'}\right) + \log\left|\frac{f(0)}{f'(0)}\right| + m(r,f'F) + T(r,f) - N(r,f)$$
$$+ q\log\frac{3q}{\delta} + \log 2 \quad \cdots (7)$$

Again by Jensen's formula

$$N\left(r,\frac{f'}{f}\right) - N\left(r,\frac{f}{f'}\right) = m\left(r,\frac{f}{f'}\right) - m\left(r,\frac{f'}{f}\right) - \log\left|\frac{f(0)}{f'(0)}\right|$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left\{\log^+\left|\frac{f(re^{i\theta})}{f'(re^{i\theta})}\right| - \log^+\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|\right\}d\theta - \log\left|\frac{f(0)}{f'(0)}\right|$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{f(re^{i\theta})}{f'(re^{i\theta})}\right|d\theta - \log\left|\frac{f(0)}{f'(0)}\right|$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta - \log|f(0)| - \frac{1}{2\pi}\int_0^{2\pi}\log|f'(re^{i\theta})|\,d\theta$$
$$+\log|f'(0)|$$

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$= \{m(r,f) - m(r,\frac{1}{f}) - \log|f(0)|\} - \{m(r,f') - m(r,\frac{1}{f'})$

$- \log|f'(0)|\}$

$= \{N(r,\frac{1}{f}) - N(r,f)\} + \{N(r,f') - N(r,\frac{1}{f'})\}$ ----- (8)

From (7) and (8) we get.

$$\sum_{\mu=1}^{q} m(r,a_\mu) + m(r,\infty) \le T(r,f) - N(r,\frac{1}{f}) + \log\frac{1}{|f(0)|}$$

$$+ m(r,\frac{f'}{f}) + N(r,\frac{1}{f}) - N(r,f) + N(r,f') - N(r,\frac{1}{f'})$$

$$+ \log\left|\frac{f(0)}{f'(0)}\right| + m(r, f'.F) + T(r,f) - N(r,f) + q\log^{+}\frac{3q}{\delta}$$

$$+ \log 2$$

$$= 2T(r,f) - \{2N(r,f) - N(r,f') + N(r,\frac{1}{f'})\}$$

$$+ m(r,\frac{f'}{f}) + m(r, f'.F) + q\log^{+}\frac{3q}{\delta} + \log 2$$

$$+ \log\frac{1}{|f'(0)|} = 2T(r,f) - N_1(r) + S(r,f).$$

i.e. $\sum_{\mu=1}^{q} m(r,a_\mu) + m(r,\infty) \le 2T(r,f) - N_1(r) + S(r,f)$

This proves the theorem.

Estimation of $S(r,f)$

Theorem Suppose that f is meromorphic and not constant in $|z| \le R_0 \le +\infty$ and $S(r,f)$ is defined as in the previous theorem.

Then

(1) if $R_0 = +\infty$

$$S(r,f) = O\{\log T(r,f)\} + O(\log r) \text{ as } r \to \infty$$

through all values of r if f is of finite order as as $r \to \infty$ outside a set E of values of r of finite linear measure, otherwise

(2) if $0 < R_0 < \infty$

$$S(r,f) = O\left\{\log^{+} T(r,f) + \log\frac{1}{R_0 - r}\right\} \text{ as } r \to R_0$$

outside a set E such that $\int_E \frac{dr}{R_0 - r} < \infty$

Further there is a point r outside E for which $\rho < r < \rho'$ provided that $0 < R_0 - \rho' < e^{-2}(R_0$

Nevanlinna's Theory of deficient values.

By the first fundamental theorem we get for a meromorphic function f

$$m(r,a) + N(r,a) = T(r,f) + O(1)$$

for any complex number a and as $r \to \infty$.

the set $\delta(a) = \delta(a,f) = \varliminf_{r \to \infty} \frac{m(r,a)}{T(r,f)}$.

$$= 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)},$$

$$\Theta(a) = \Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a)}{T(r,f)},$$

$$\theta(a) = \theta(a;f) = \varliminf_{r\to\infty} \frac{N(r,a) - \overline{N}(r,a)}{T(r,f)},$$ where

$\overline{N}(r,a)$ is the counting function of distinct a-point of f.

Evidently, given $\varepsilon > 0$, we get for all large values of r

$$N(r,a) - \overline{N}(r,a) > \{\theta(a) - \varepsilon\} T(r,f)$$

$$N(r,a) < \{1 - \delta(a) + \varepsilon\} T(r,f)$$

and hence $\overline{N}(r,a) < \{1 - \delta(a) - \theta(a) + 2\varepsilon\} T(r,f)$

so that $\Theta(a) \geq \delta(a) + \theta(a)$

The quantity $\delta(a,f)$ is called the <u>deficiency</u> of the value a and $\Theta(a,f)$ is called the <u>index of multiplicity</u>. Clearly $\delta(a,f)$ is positive only if there are relatively few roots of the equation $f(z) = a$, while $\theta(a)$ is positive if there are relatively many multiple roots.

<u>Lemma A</u> Suppose that f is meromorphic and not constant in $|z| < R_0$. If $S(r,f)$ is defined as in the second fundamental theorem then

$$\frac{S(r,f)}{T(r,f)} \to 0 \text{ as } r \to R_0 \quad \cdots (1)$$

with the following provisions:

(a) (1) holds without restrictions if $R_0 = +\infty$ and if f is meromorphic of finite order in the plane;

(b) If f has infinite order in the plane (1) still holds as $r \to \infty$ outside a certain exceptional set E_0 of finite length. Here E_0 depends on ~~the~~ f but not on a_ν or q.

(c) If $R_0 < +\infty$ and $\limsup_{r\to R_0} \frac{T(r,f)}{\log\{1/(R_0-r)\}} = +\infty$

then (1) holds as $r \to R_0$ through a sequence $\{r_n\}$ which depends on f but not on the a_ν or on q.

<u>Definition</u> A meromorphic function f is said to be admissible in $|z| < R_0$ if either $R_0 < \infty$ and

$$\limsup_{r\to R_0} \frac{T(r,f)}{\log\{1/(R_0-r)\}} = +\infty$$

or $R_0 = +\infty$ and f is nonconstant.

By the above lemma we see that for an admissible function f, $\frac{S(r,f)}{T(r,f)} \to 0$ as $r \to R_0$ through a sequence of values.

Nevanlinna's Theorem on deficient values.

Theorem Suppose that f is admissible in $|z| < R_0$ — Then the set of values a for which $\Theta(a) > 0$ is countable, and we have, on summing over all such values a

$$\sum_a \{ \delta(a) + \theta(a) \} \le \sum_a \Theta(a) \le 2.$$

Proof We choose a sequence $r_n \to R_0$ such that $S(r_n, f) = o\{T(r_n, f)\}$.

From the second fundamental theorem we get

$$m(r_n, \infty) + \sum_{\nu=1}^{q} m(r_n, a_\nu) \le 2T(r_n, f) - N_1(r_n) + S(r_n, f) \cdots (1)$$

Adding $N(r_n, \infty) + \sum_{\nu=1}^{q} N(r_n, a_\nu)$ to both sides of (1) we get by the first fundamental ~~theory~~ theorem

$$(q+1)\,T(r_n, f) < \sum_{\nu=1}^{q} N(r_n, a_\nu) + N(r_n, \infty) - N_1(r_n) + \{2+o(1)\}\, T(r_n, f),$$

where $N_1(r_n) = N(r_n, \frac{1}{f'}) + 2N(r_n, f) - N(r_n, f')$

i.e., $\{q - 1 + o(1)\}\, T(r_n, f) < \sum_{\nu=1}^{q} N(r_n, a_\nu) + \bar{N}(r_n, \infty) - N(r_n, \frac{1}{f'})$,

because $N(r_n, f') - N(r_n, f) = \bar{N}(r_n, f)$.

Now a root of the equation $f(z) = a_\nu$ of multiplicity p is also a zero of multiplicity $p-1$ of $f'(z)$ and so contributes only 1 to $n(t, a_\nu) - n(t, \frac{1}{f'})$.

Thus we may write our inequality as

$$\{q - 1 + o(1)\}\, T(r_n, f) \le \sum_{\nu=1}^{q} \bar{N}(r_n, a_\nu) + \bar{N}(r_n, \infty) - N_0(r_n, \frac{1}{f'}), \cdots (2)$$

where $N_0(r, \frac{1}{f'})$ refers to those zeros of f' which occur at points other than roots of the eqn $f(z) = a_\nu$ $(\nu = 1, 2, \ldots, q)$. Ignoring this latter term and dividing by $T(r_n, f)$ we deduce that

$$\sum_{\nu=1}^{q} \limsup_{r \to R_0} \frac{\bar{N}(r, a_\nu)}{T(r,f)} + \limsup_{r \to R_0} \frac{\bar{N}(r, \infty)}{T(r,f)}$$

$$\ge \limsup_{r \to R_0} \frac{\sum_{\nu=1}^{q} \bar{N}(r, a_\nu) + \bar{N}(r, \infty)}{T(r,f)} \ge q - 1$$

i.e., $\sum_{\nu=1}^{q}\{1-\Theta(a_\nu;f)\}+1-\Theta(\infty,f)\geq q-1$

i.e., $\sum_{\nu=1}^{q}\Theta(a_\nu;f)+\Theta(\infty,f)\leq 2$

The result shows that $\Theta(a;f)>\frac{1}{N}$ for at most $2N-1$ distinct finite values a. Thus the values a for which $\Theta(a)>0$ may be arranged in a sequence, in order of decreasing $\Theta(a,f)$, by taking first those for which $\Theta(a;f)=1$, then those, if any, for which $\Theta(a,f)>\frac{1}{2}$ then those of the remainder for which $\Theta(a,f)>\frac{1}{3}$ and so on.

If $\{a_\nu\}$ is the resulting sequence together with $a_0=\infty$, we deduce that

$$\sum_{\nu=0}^{q}\Theta(a_\nu;f)\leq 2$$

for any finite q and hence if the sequence $\{a_\nu\}$ is infinite, we deduce that

$$\sum_{\nu=0}^{\infty}\Theta(a_\nu;f)\leq 2.$$

This proves the theorem.

Deficient Functions

Theorem If f is meromorphic and admissible in $|z|<R_0$ and $a_1(z), a_2(z), a_3(z)$ are distinct meromorphic functions satisfying for $\nu=1,2$ and 3

$$T(r, a_\nu(z))=o\{T(r,f)\} \text{ as } r\to R_0$$

Then $$\{1+o(1)\}T(r,f)\leq\sum_{\nu=1}^{3}\bar{N}\left(r,\frac{1}{f-a_\nu}\right)+S(r,f)$$

as $r\to R_0$ where $S(r,f)=o\{T(r,f)\}$ as $r\to R_0$ at least along a sequence of values of r.

Proof Let $\phi(z)=\dfrac{f(z)-a_1(z)}{f(z)-a_3(z)}\cdot\dfrac{a_2(z)-a_3(z)}{a_2(z)-a_1(z)}$.

By the second f. Theorem we get

$$m(r,\infty;\phi)+m(r,0;\phi)+m(r,1;\phi)$$
$$\leq 2T(r,\phi)-N_1(r,\phi)+S(r,\phi).$$

Adding $N(r,\infty;\phi)+N(r,0;\phi)+N(r,1;\phi)$ to both sid[es] we get by the first fundamental theorem.

$$T(r,\phi)\leq N(r,\infty;\phi)+N(r,0;\phi)+N(r,1;\phi)-N_1(r,\phi)+$$

where $N_1(r,\phi)=N(r,0;\phi')+2N(r,\phi)-N(r,\phi')$

So, $T(r,\phi)\leq N(r,0;\phi)+N(r,1;\phi)+\bar{N}(r,\infty;\phi)-N$

$+ S(r, \varphi')$,

because $N(r, \varphi') - N(r, \varphi) = \bar{N}(r, \infty; \varphi)$

Now a root of the eqn $\varphi(z) = a$ of multiplicity p is also a zero of $\varphi'(z)$ with multiplicity $p-1$ and so $N(r, 0; \varphi) + N(r, 1; \varphi) - N(r, 0; \varphi')$

$$\le \bar{N}(r, 0; \varphi) + \bar{N}(r, 1; \varphi).$$

Therefore we get

$$T(r, \varphi) \le \bar{N}(r, 0; \varphi) + \bar{N}(r, 1; \varphi) + \bar{N}(r, \infty; \varphi) + S(r, \varphi) \quad \text{---- (1)}$$

Now, $T(r, f) \le T(r, f - a_3) + T(r, a_3) + O(1)$

$$= T\left(r, \frac{1}{f - a_3}\right) + o\{T(r, f)\}$$

$$\le T\left(r, \frac{a_3 - a_1}{f - a_3}\right) + T\left(r, \frac{1}{a_3 - a_1}\right) + o\{T(r, f)\}$$

$$= T\left(r, \frac{a_3 - a_1}{f - a_3}\right) + T(r, a_3 - a_1) + o\{T(r, f)\}$$

$$\le T\left(r, \frac{a_3 - a_1}{f - a_3}\right) + o\{T(r, f)\}$$

$$= T\left(r, 1 + \frac{a_3 - a_1}{f - a_3} - 1\right) + o\{T(r, f)\}$$

$$\le T\left(r, 1 + \frac{a_3 - a_1}{f - a_3}\right) + o\{T(r, f)\}$$

$$= T\left(r, \frac{f - a_1}{f - a_3}\right) + o\{T(r, f)\}$$

Again $T\left(r, \frac{a_2 - a_1}{a_2 - a_3}\right)$

$$\le T(r, a_2 - a_1) + T(r, a_2 - a_3) + O(1)$$

$$\le T(r, a_1) + 2T(r, a_2) + T(r, a_3) + O(1)$$

$$= o\{T(r, f)\}.$$

Now, $\{1 + o(1)\} T(r, f) \le T\left(r, \frac{f - a_1}{f - a_3}\right)$

$$= T\left(r, \varphi \cdot \frac{a_2 - a_1}{a_2 - a_3}\right)$$

$$\le T(r, \varphi) + T\left(r, \frac{a_2 - a_1}{a_2 - a_3}\right)$$

$$= T(r, \varphi) + o\{T(r, f)\}$$

i.e., $\{1 + o(1)\} T(r, f) \le T(r, \varphi)$ ----- (2) as $r \to R$

Also we see that

$$T(r, \varphi) \le T\left(r, \frac{f - a_1}{f - a_3}\right) + T\left(r, \frac{a_2 - a_3}{a_2 - a_1}\right)$$

$$\le 2T(r, f) + o\{T(r, f)\}$$

$$= \{2 + o(1)\} T(r, f).$$

Therefore $S(r, \varphi) = S(r, f)$.

Finally the equations $\varphi(z) = 0, 1, \infty$ have roots only if either $f(z) - a_\nu(z) = 0$ for $\nu = 1, 2$ or 3 or if two of the functions a_ν become equal. Thus

... other hand, each common root of equation $f_j(z) = a$ $(j=1,2)$ is a pole of ... and so

$$\ldots(r) \le \overline{N}\left(r, \infty; \frac{1}{f_1 - f_2}\right)$$

$$\le T\left(r, \frac{1}{f_1 - f_2}\right) \quad \text{---- (2)}$$

... we get from (1) and (2)

$$\ldots\left(r, \frac{1}{f_1 - f_2}\right) \le \left\{\frac{2}{3} + o(1)\right\} T\left(r, \frac{1}{f_1 - f_2}\right)$$

... $\to \infty$ through a sequence of values. ... is a contradiction and so the Theorem ...ved.

... Theory

... a meromorphic and not constant in the ... plane. We call an error term and denote by ... any quantity satisfying

$$S(r,f) = o\{T(r,f)\}$$

as $r \to \infty$, possibly outside a set of r of finite linear measure.

Let $a = a(z)$ be a meromorphic function in the complex plane. We call the function $a = a(z)$ a small function of f if $T(r,a) = S(r,f)$.

<u>Theorem</u> Let l be positive integer and

$$\psi(z) = \sum_{\nu=0}^{l} a_\nu(z) f^{(\nu)}(z),$$

where $a_0(z), a_1(z), \ldots, a_l(z)$ are small functions of f.

Then (i) $m\left(r, \frac{\psi}{f}\right) = S(r,f)$

and (ii) $T(r,\psi) \le (l+1)\, T(r,f) + S(r,f)$.

<u>Proof</u> From the second fundamental Theorem we get

$$S(r,f) = m\left(r, \frac{f'}{f}\right) + m\left\{r, \sum_{\nu=1}^{q} \frac{f'}{f - a_\nu}\right\} + q \log^+ \frac{3q}{\delta} + \log 2 + \log \frac{1}{|f'(0)|}$$

where $a_1, a_2, \ldots, a_q$ are distinct complex numbers and we assume that $f(0) \ne 0, \infty$, $f'(0) \ne 0$.

Again by Lemma A we see that

$$\lim_{r \to \infty} \frac{S(r,f)}{T(r,f)} = 0$$

except possibly a set of finite linear measure.

Therefore we get $\boxed{m\left(r, \frac{f'}{f}\right) = S(r,f)}$

Suppose we have already proved that

$$m\left(r, \frac{f^{(l)}}{f}\right) = S(r,f)$$

for some l. We deduce that

$m(r, f^{(l)}) \le m(r, \frac{f^{(l)}}{f}) + m(r, f) = m(r, f) + S(r, f)$.

Also if $f(z)$ has a pole of multiplicity k at z_0, $f^{(l)}(z)$ has a pole of multiplicity

$k + l \le (l+1)k$

there, so that

$N(r, f^{(l)}) \le (l+1) N(r, f)$

Thus by addition we deduce that

$$T(r, f^{(l)}) = m(r, f^{(l)}) + N(r, f^{(l)})$$
$$\le m(r, f) + (l+1) N(r, f) + S(r, f)$$
$$\le (l+1) T(r, f) + S(r, f),$$

We deduce that

$$m(r, \frac{f^{(l+1)}}{f^{(l)}}) = S(r, f^{(l)}) = o\{T(r, f^{(l)})\} = o\{T(r, f)\}$$

as $r \to \infty$ outside a set of finite linear measure.

Thus $m(r, \frac{f^{(l+1)}}{f}) \le m(r, \frac{f^{(l+1)}}{f^{(l)}}) + m(r, \frac{f^{(l)}}{f})$

$$= S(r, f) + S(r, f) = S(r, f).$$

This completes the inductive proof in case $\psi(z) = f^{(l)}(z)$.

To deal with the general case note that

$$m(r, \frac{\psi}{f}) \le \sum_{\nu=0}^{l} m(r, \frac{a_\nu f^{(\nu)}}{f}) + \log(l+1)$$

$$\le \sum_{\nu=0}^{l} [m(r, a_\nu) + m(r, \frac{f^{(\nu)}}{f})] + \log(l+1).$$

$$\le \sum_{\nu=0}^{l} S(r, f) + O(1) = S(r, f).$$

This proves (i).

Further we have

$$m(r, \psi) \le m(r, \frac{\psi}{f}) + m(r, f)$$
$$= m(r, f) + S(r, f) \quad \cdots\cdots (1)$$

Also if f has a pole of multiplicity p at z_0 and a_ν's have poles of multiplicity at most q there, then ψ has a pole of multiplicity at most

$$p + l + q \le (l+1)p + q = (l+1)p + q$$

at z_0. Thus

$$N(r, \psi) \le (l+1) N(r, f) + \sum_{\nu=0}^{l} N(r, a_\nu)$$
$$= (l+1) N(r, f) + S(r, f) \cdots\cdots (2).$$

Therefore from (1) and (2) we get (ii). This proves the theorem.

Milloux's Basic Result.

Theorem Suppose that f is nonconstant and meromorphic in the plane and $\psi(z) = \sum_{\nu=0}^{l} a_\nu(z) f^{(\nu)}(z)$, where $T(r, a_\nu) = S(r, f)$ for $\nu = 0, 1, 2, \ldots, l$.

Then $T(r, f) \le \overline{N}(r, f) + N(r, 0; f) + \overline{N}(r, 1; \psi) - N_0(r, 0; \psi')$
$+ S(r, f)$,

$$\bar{N}(r, \infty; \varphi) + \bar{N}(r, 0; \varphi) + \bar{N}(r, 1; \varphi)$$

$$\le \sum_{\nu=1}^{3} \bar{N}\left(r, \frac{1}{f-a_\nu}\right) + \bar{N}\left(r, \frac{1}{a_1-a_2}\right) + \bar{N}\left(r, \frac{1}{a_2-a_3}\right) + \bar{N}\left(r, \frac{1}{a_3-a_1}\right)$$

$$\le \sum_{\nu=1}^{3} \bar{N}\left(r, \frac{1}{f-a_\nu}\right) + o\{T(r,f)\} \quad \cdots\cdots (3)$$

as $r \to R_0$

Now from (1), (2) and (3) we get.

$$\{1+o(1)\} T(r,f) \le \sum_{\nu=1}^{3} \bar{N}\left(r, \frac{1}{f-a_\nu}\right) + S(r,f)$$

This proves the theorem.

Nevanlinna's five-point uniqueness theorem

Theorem Suppose that $f_1(z)$, $f_2(z)$ are meromorphic in the complex plane and let $E_j(a)$ be the set of points z such that $f_j(z) = a$ $(j=1,2)$. If $E_1(a) = E_2(a)$ for five distinct values of f then $f_1 \equiv f_2$ or f_1, f_2 are both constant.

Proof We suppose that f_1, f_2 are neither both constant nor identical.

Let a_1, a_2, a_3, a_4, a_5 be distinct and such that the sets $E_1(a_\nu) = E_2(a_\nu)$ for $\nu = 1, 2, \ldots, 5$.

We write

$$N_\nu(r) = \bar{N}(r, a_\nu; f_1) = \bar{N}(r, a_\nu; f_2)$$

for $\nu = 1, 2, \ldots, 5$.

If f_1 is constant, f_2 omits at least four and so f_2 is also constant, which is excluded by our hypotheses.

Now by the second fundamental theorem we get as $r \to \infty$ through a suitable sequence of values

$$\{4+o(1)\} T(r, f_j) \le \sum_{\nu=1}^{5} N_\nu(r) + \bar{N}(r, f_j), \quad \text{for } j = 1, 2$$

So $\{3+o(1)\} T(r, f_j) \le \sum_{\nu=1}^{5} N_\nu(r)$,

as $r \to \infty$ through a suitable sequence of values.

Since f_1, f_2 are not identical, we get as $r \to \infty$ through this sequence

$$T\left(r, \frac{1}{f_1-f_2}\right) = T(r, f_1-f_2) + O(1)$$

$$\le T(r, f_1) + T(r, f_2) + O(1)$$

$$\le \left\{\frac{2}{3} + o(1)\right\} \sum_{\nu=1}^{5} N_\nu(r) \quad \cdots (1)$$

where in $N_0(r, 0; \psi')$ only zeros of ψ' not corresponding to the repeated roots of $\psi(z) = 1$ are to be considered.

Proof Applying the second fundamental theorem to $\psi(z)$ we obtain

$$m(r, \psi) + m(r, 0; \psi) + m(r, 1; \psi) \le 2T(r, \psi) - N_1(r, \psi) + S(r, \psi) \quad \cdots (1)$$

Now

$$2T(r, \psi) - N_1(r, \psi) = m(r, \psi) + m(r, 1; \psi) + N(r, \psi) + N(r, 1; \psi) - N(r, 0; \psi') - 2N(r, \psi) + N(r, \psi') + O(1) \quad \cdots (2)$$

Again at a pole of ψ of multiplicity p, ψ' has a pole of multiplicity $p+1$. These poles occur only at poles of f or of $a_\nu(z)$. Thus

$$N(r, \psi') - N(r, \psi) = \overline{N}(r, \psi) \le \overline{N}(r, f) + \sum_{\nu=0}^{l} \overline{N}(r, a_\nu)$$

$$= \overline{N}(r, f) + S(r, f).$$

Further, at a zero of $\psi - 1$ of multiplicity p, ψ' has a zero of multiplicity $p-1$, so that

$$N(r, 1; \psi) - N(r, 0; \psi') = \overline{N}(r, 1; \psi) - N_0(r, 0; \psi')$$

Since $T(r, \psi) \le (l+1)\, T(r, f) + S(r, f)$, it follows that $S(r, \psi) = o\{T(r, \psi)\} = o\{T(r, f)\}$ outside a set of finite linear measure, so that

$$S(r, \psi) = S(r, f).$$

Thus (1) and (2) yield

$$m(r, 0; \psi) \le \overline{N}(r, f) + \overline{N}(r, 1; \psi) - N_0(r, 0; \psi') + S(r, f) \quad \cdots (3)$$

Again we see that

$$T(r, f) = m(r, 0; f) + N(r, 0; f) + O(1)$$

$$\le m(r, 0; \psi) + m\left(r, \frac{\psi}{f}\right) + N(r, 0; f) + O(1)$$

$$\le m(r, 0; \psi) + N(r, 0; f) + S(r, f) \quad \cdots (4)$$

Now from (3) and (4) we get

$$T(r, f) \le \overline{N}(r, f) + N(r, 0; f) + \overline{N}(r, 1; \psi) - N_0(r, 0; \psi') + S(r, f)$$

This proves the Theorem.

<u>Theorem</u> If f is meromorphic and transcendental in the plane and has only a finite number of zeros and poles, then

$$\psi(z) = \sum_{\nu=0}^{l} a_\nu(z) f^{(\nu)}(z)$$

assumes every finite complex value except possibly zero infinitely often or else ψ is identically constant.

<u>Proof</u> Let $\psi_1(z) = \frac{1}{a}\psi(z)$, where $a\,(\neq 0)$ is a f[illegible] complex number.

Further suppose that ψ is nonconstant. Then ψ_1 is also nonconstant. Since f has only finite number of zeros and poles, we get

$\overline{N}(r,f) = O(\log r)$ and $N(r,0,f) = O(\log r)$.

So by the preceding theorem we get

$$T(r,f) \le \overline{N}(r,1;\psi_1) + O(\log r) + S(r,f)$$

$$= \overline{N}(r,a;\varphi) + S(r,f),$$

because f is transcendental and so $\frac{\log r}{T(r,f)} \to 0$

as $r \to \infty$.

Therefore φ assumes the value 'a' infinitely many times.

Subtracting (4) from (3) we get

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_\rho} \left[\frac{f(w)}{w-z} - \frac{f(w)}{w-z^*}\right] dw \quad \cdots (5)$$

$$= \frac{1}{2\pi i} \int_{\gamma_\rho} f(w) \left[\frac{1}{w-z} - \frac{1}{w-z^*}\right] dw \quad \cdots (5)$$

Let $w - z = (w - z_0) - (z - z_0) = \rho e^{i\varphi} - re^{i\theta}$ and

$w - z^* = (w - z_0) - (z^* - z_0) = \rho e^{i\varphi} - \frac{\rho^2}{r} e^{i\theta}$.

Then (5) becomes

$$f(z) = \frac{1}{2\pi i} \int_0^{2\pi} f(w) \left[\frac{1}{\rho e^{i\varphi} - re^{i\theta}} - \frac{1}{\rho e^{i\varphi} - \frac{\rho^2}{r} e^{i\theta}}\right] i\rho e^{i\varphi} d\varphi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\varphi}) \left[\frac{\rho}{\rho - re^{i(\theta-\varphi)}} + \frac{re^{i(\varphi-\theta)}}{\rho - re^{i(\varphi-\theta)}}\right] d\varphi.$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\varphi}) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)} d\varphi \quad \cdots (6)$$

Since $f(z) = u(r,\theta) + iv(r,\theta)$ and $f(z_0 + \rho e^{i\varphi}) = u(\rho,\varphi)$ ~~+ i v(ρ,φ)~~ $+ iv(\rho,\varphi)$, equating the real and imaginary parts of (6) we get (1) and (2).

Note. The function $\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}$ is called Poisson's Kernel.

Ex Show that Poisson's kernel is a harmonic function in $|z - z_0| < \rho$.

Soln: Let $f(z) = \frac{\rho e^{i\varphi} + (z - z_0)}{\rho e^{i\varphi} - (z - z_0)}$, which is analytic in the disc $|z - z_0| < \rho$.

We put $z - z_0 = re^{i\theta}$ and denote the real and imaginary parts of $f(z)$ in polar coordinates by $u(r,\theta)$ and $v(r,\theta)$. Then

$$u(r,\theta) + iv(r,\theta) = \frac{\rho e^{i\varphi} + re^{i\theta}}{\rho e^{i\varphi} - re^{i\theta}}$$

$$= \frac{(\rho e^{i\varphi} + re^{i\theta})(\rho e^{-i\varphi} - re^{-i\theta})}{(\rho e^{i\varphi} - re^{i\theta})(\rho e^{-i\varphi} - re^{-i\theta})}$$

$$= \frac{\rho^2 - r^2 + \rho r e^{i(\theta-\varphi)} - \rho r e^{-i(\theta-\varphi)}}{\rho^2 + r^2 - \rho r e^{i(\theta-\varphi)} - \rho r e^{-i(\theta-\varphi)}}$$

$$= \frac{\rho^2 - r^2 + 2i\rho r \sin(\theta - \varphi)}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}$$

So, $u(r,\theta) = \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}$ and $v(r,\theta) = \frac{2\rho r \sin(\theta - \varphi)}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}$

are conjugate harmonic fns. in $|z - z_0| < \rho$, being the real and imaginary parts of an analytic fn. Hence

$u(r,\theta)$, the Poisson kernel, is harmonic in $|z-z_0|<\rho$.

Ex Show that $\dfrac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos(\theta-\varphi)} = 1+2\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^n \cos n(\theta-\varphi)$,

$$\frac{2\rho r\sin(\theta-\varphi)}{\rho^2+r^2-2\rho r\cos(\theta-\varphi)} = 2\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^n \sin n(\theta-\varphi) \quad \text{in } |z-z_0|<\rho.$$

Soln We see that

$$\frac{\rho e^{i\varphi}+(z-z_0)}{\rho e^{i\varphi}-(z-z_0)} = \frac{(z-z_0)-\rho e^{i\varphi}+2\rho e^{i\varphi}}{\rho e^{i\varphi}-(z-z_0)} = -1+\frac{2\rho e^{i\varphi}}{\rho e^{i\varphi}-(z-z_0)}$$

$$= -1+\frac{2}{1-\frac{z-z_0}{\rho e^{i\varphi}}} = -1+2\left(1-\frac{z-z_0}{\rho e^{i\varphi}}\right)^{-1}$$

$$= -1+2\left[1+\frac{z-z_0}{\rho e^{i\varphi}}+\left(\frac{z-z_0}{\rho e^{i\varphi}}\right)^2+\cdots\right]$$

$$= 1+2\sum_{n=1}^{\infty}\frac{(z-z_0)^n}{\rho^n}e^{-in\varphi}$$

Now we put $z-z_0=re^{i\theta}$ so that

$$\frac{\rho e^{i\varphi}+(z-z_0)}{\rho e^{i\varphi}-(z-z_0)} = 1+2\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^n e^{in(\theta-\varphi)},$$

Equating the real and imaginary parts we get,

$$\frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos(\theta-\varphi)} = 1+2\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^n \cos n(\theta-\varphi)$$

$$\frac{2\rho r\sin(\theta-\varphi)}{\rho^2+r^2-2\rho r\cos(\theta-\varphi)} = 2\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^n \sin n(\theta-\varphi) \quad \text{in } |z-z_0|<\rho.$$

Mean value property

Let f be an analytic fn in a domain D. Let $z_0\in D$ and let Γ be a circle with centre z_0 and contained in D.

We take the circle $|z-z_0|=r$ as Γ and write $z=z_0+re^{i\theta}$, $0\le\theta\le 2\pi$. By Cauchy's Integral formula we get

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)\,dz}{z-z_0}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta,$$ for all values of r for which Γ lies within D.

This result is known as mean value property theorem of Gauss.

It thus follows that for any closed disc contained in D in which f is analytic, the value of f at the centre of the disc equals the mean of the values of f on the boundary of the disc.

Defn We shall say a real or complex valued cont mean value property (in short M-V.P.) if for any compact disc contained in D, the value of f at the centre of the disc is equal to the mean of its values on the boundary of the disc.

We can say that any analytic fn has the m-v-p.

<u>Th.</u> Any harmonic fn defined in a domain D has the m-v-p.

<u>Proof</u> Let $u(x,y)$ be a harmonic fn given in a domain D and let S be a closed disc contained in D. Then there exists an analytic fn f, say, in D whose real part is u.

Now the value of f at the centre of S is equal to the mean of f on the boundary of the disc and by taking the real parts we see that the value of $u(x,y)$ at the centre of S is equal to the mean value on the boundary of the disc. This proves the theorem.

<u>Note</u> In fact we get

$$u(x_0,y_0)=\frac{1}{2\pi}\int_0^{2\pi} u(x_0+r\cos\theta,\ y_0+r\sin\theta)\,d\theta,$$ where $z_0=x_0+iy_0$ the centre of S and r is the radius of S.

<u>Ex</u> When $|z|\le R$, the fn f is regular and satisfies the inequality $|f(z)|\ge 1$. Show by applying Poisson's formula to $\log f(z)$ that if $|z|\le kR$ where $k<1$ then $|f(z)|\le |f(0)|^{\frac{1+k}{1-k}}$

<u>Soln</u> Poisson's formula states that if f is regular in $|z|\le R$ and $u(r,\theta)$ its real part, then for $0\le r<R$

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)\,u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}\,d\phi .$$

Now by condition of the problem $|f(z)|\ne 0$ in $|z|\le R$ and hence $\log f(z)$ is regular within and on $|z|=R$ and the real part of $\log f(z)$ is $\log|f(z)|$.

Let $z=re^{i\theta}$, where $r<R$, and $k=\frac{r}{R}<1$. Hence applying Poisson's formula to $\log f(z)$ we have

$$\log|f(re^{i\theta})|=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)\log|f(Re^{i\phi})|}{R^2+r^2-2rR\cos(\theta-\phi)}\,d\phi \quad \text{---}(1)$$

From (1) we get for $r=0$

$$\log|f(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\phi})|\,d\phi \quad \text{---}(2)$$

Again from (1) we get

$$\log|f(re^{i\theta})|\le\frac{1}{2\pi}\,\frac{R^2-r^2}{\ldots}\int^{2\pi}\log|f(Re^{i\phi})|\,d\phi .$$

$$= \frac{1}{2\pi} \frac{R+r}{R-r} \int_0^{2\pi} \log |f(Re^{i\phi})| \, d\phi$$

$$= \frac{R+r}{R-r} \log |f(0)|.$$

i.e. $|f(re^{i\theta})| \le |f(0)|^{\frac{R+r}{R-r}} = |f(0)|^{\frac{1+r/R}{1-r/R}}$

i.e. $|f(re^{i\theta})| \le |f(0)|^{\frac{1+k}{1-k}}$

i.e. $|f(z)| \le |f(0)|^{\frac{1+k}{1-k}}$, where $z = re^{i\theta}$.

Since the maximum of $|f(z)|$ on $|z| = r$ does not exceed $|f(0)|^{\frac{1+k}{1-k}}$, by the maximum modulus principle same inequality holds at any point z such that $|z| < r$.

Hence $|f(z)| \le |f(0)|^{\frac{1+k}{1-k}}$ where $|z| \le r = kR$, $R < 1$.

<u>Ex</u> Let f be a fn regular in the closed disc $|z| \le R$ and let $u(r,\theta)$ be its real part. If $u(r,\theta) \ge 0$ prove that the inequalities.

$$\frac{R-r}{R+r} u(0,0) \le u(r,\theta) \le \frac{R+r}{R-r} u(0,0) \quad (0 \le r < R)$$

<u>Soln.</u> Using Poisson's formula we get.

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2-r^2)\, u(R,\phi)}{R^2 - 2rR\cos(\theta-\phi) + r^2} \, d\phi \quad \text{---(1)}$$

Now $\frac{R+r}{R-r} = \frac{R^2-r^2}{(R-r)^2} \ge \frac{R^2-r^2}{R^2 - 2rR\cos(\theta-\phi) + r^2} \ge \frac{R^2-r^2}{(R+r)^2}$

$$= \frac{R-r}{R+r} \quad (r < R).$$

Since $u(r,\theta) \ge 0$ throughout the disc $|z| < R$, we obtain

$$\frac{R-r}{R+r} \cdot \frac{1}{2\pi} \int_0^{2\pi} u(R,\phi)\, d\phi \le \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2-r^2)\, u(R,\phi)}{R^2 - 2rR\cos(\theta-\phi) + r^2} \, d\phi$$

$$\le \frac{R+r}{R-r} \frac{1}{2\pi} \int_0^{2\pi} u(R,\phi)\, d\phi$$

But $u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} u(R,\phi)\, d\phi$ [by putting $r=0$, $\theta=0$ in (1)].

So. $\frac{R-r}{R+r} u(0,0) \le u(r,\theta) \le \frac{R+r}{R-r} u(0,0) \quad (r < R)$

<u>Note</u> :- The above inequality is known as Harnack's inequality. This is valid for all positive harmonic functions. Putting $r = R/2$ we obtain $\frac{1}{3} u(0,0) \le u(r,\theta) < 3u(0,0)$.

<u>Ex</u> Find the analytic fn $f(z) = u(x,y) + iv(x,y)$ given that $u(x,y) = x^2 - y^2 + 2$.

<u>Soln</u> If we can find out the harmonic conjugate $v(x,y)$ of $u(x,y) = x^2 - y^2 + 2$ then $f = u + iv$ will be an analytic fn.

Now $v_y = u_x = 2x$ and $-v_x = u_y = -2y$. So $v = \int v_x \, dx$

$= \int 2y \, dx + \phi(y) = 2xy + \phi(y)$. Hence $v_y = 2x + \phi'(y)$

and so $2x + \phi'(y) = 2x$ i.e. $\phi'(y) = 0$ i.e. $\phi(y) = C$, where C is a real constant. Therefore $v(x,y) = 2xy + C$, and so the required analytic function is

$$f(z) = u + iv = x^2 - y^2 + 2 + i(2xy + c) = z^2 + 2 + ic$$

Theorem: Let $u(x,y)$ be a harmonic f^z in a domain G, with harmonic conjugate $v(x,y)$, let z_0 be an arbitrary (finite) pt of G and let $\Delta = \Delta(z_0)$ be the distance between z_0 and the boundary of G. Then $u(x,y)$ and $v(x,y)$ have expansions of the form:

$$u(x,y) = u(r,\theta) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n \quad \text{---} (1)$$

$$v(x,y) = v(r,\theta) = \beta_0 + \sum_{n=1}^{\infty} (\beta_n \cos n\theta + \alpha_n \sin n\theta) r^n \quad \text{---} (2)$$

on the disc $|z - z_0| < \Delta$, where $z - z_0 = re^{i\theta}$.

Proof: Since $u(x,y)$, $v(x,y)$ are conjugate h-f^n. $f(z) = u(x,y) + iv(x,y)$ is analytic in G. Then we can expand $f(z)$ in Taylor series about z_0 to get

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{---} (3) \quad \text{where } |z - z_0| < \Delta.$$

Now we put $a_n = \alpha_n + i\beta_n$ and $z - z_0 = re^{i\theta}$ so that (3) becomes.

$$f(z) = \sum_{n=0}^{\infty} (\alpha_n + i\beta_n) r^n e^{i\theta n}$$

$$= \sum_{n=0}^{\infty} (\alpha_n + i\beta_n) r^n (\cos n\theta + i \sin n\theta)$$

$+ \alpha_n \sin n\theta) r^n \}$.

Equating the real and imaginary parts of both sides we get (1) and (2). This proves the theorem.

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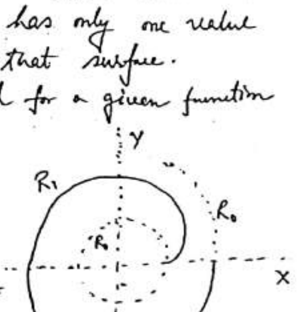
Riemann Surface

A Riemann surface is a generalisation of the z-plane into a surface of more than one sheet such that a multiple valued function has only one value corresponding to each point on that surface.

Once such a function is deviced for a given function that f^n is a single valued of function of points on the surface and the theory of single valued function can be used in dealing with the f^n. The complexities arising because the f^n is multiple value are thus removed by means of a geometrical device.



Riemann surface for the f^n $w = f(z) = z^{1/2}$

The f^n $w = z^{1/2} = r^{1/2}(\cos\frac{\theta}{2} + i\sin\theta/2)$ has two values corresponding to each pt. except the origin. Let the plane be replaced by two sheets, R_0 and R_1, each of which is cut along the positive x axis, with R_1 placed infront of R_0. Join the lower edge of the slit in R_0 to the upper edge of the slit in R_1, and the lower edge of the slit in R_1 to the upper edge of the slit in R_0. The two sheets therefore cross each other at the ...

As the point z describes a continuous circuit ... the origin on that surface, the angle θ grows from 0 to 2π and then the pt. passes from the sheet R_0 to the sheet R_1, where θ grows from 2π to 4π. As the pt. moves still further, it passes back to the sheet R_0 where the values of θ, vary either from 4π to 6π or from 0 to 2π, a choice that does not affect the value of the f^n $z^{1/2}$ and so on. The f^n is a single valued f^n of points on this surface except that some more artificial device is needed to distinguish between points of the two sheets along the curve.

The image of the sheet R_0 of this Riemann surface for the f^n $z^{1/2}$ is the upper half of the w-plane since $w = \rho e^{i\varphi} = r^{1/2}e^{i\theta/2}$, and $0 \le \theta/2 \le \pi$ on R_0. The image of the sheet is the lower half of the w-plane.

The simplest many valued function are these which satisfied an algebraic equation, say

$$F(\omega, z) = \omega^n + a_1(z)\,\omega^{n-1} + a_2(z)\,\omega^{n-2} + \cdots + a_{n-1}(z)\,\omega + a_n(z) = 0$$

Solving for ω we get. $\omega = f_1(z), f_2(z), \cdots, f_n(z)$ which are called branches of the many valued function $F(\omega, z)$. There will be certain points in general, at which two or more branches coincide. These points are called branch points. Thus in ~~analy~~ analogy to the theory of eq^n, a simplest many valued fn has a branch point if the two equations $F(\omega, z) = 0$ and $\frac{\partial F}{\partial \omega} = 0$ are consistent. Now eleminating ω between these two equations we get an eq^n in z, say $\phi(z) = 0$, determining the branch point of the many valued function. Putting $n = 2$ we have $\omega^2 + a_1(z)\omega + a_2(z) = 0$ and taking $a_1(z) = 0$, we have $\omega^2 + a_2(z) = 0$. We shall mainly consider the special type of the simplest many valued function. Thus for the multiple valued function $\omega^2 = f(z)$, the branches are $\omega_1 = \sqrt{f(z)}$, $\omega_2 = -\sqrt{f(z)}$ and when the two branches are equal, $f(z) = 0$ which determines the branch points.

Multiple value functions, branch point and branch cut

Def^n If only one value of ω corresponds to each value of z, then we say that ω is a single-valued f^n of z.

If more than one value of ω corresponds to each value of z, we say that ω is a multiple-valued or many-valued f^n of z.

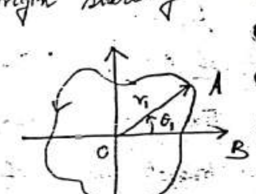
A multiple-valued f^n can be considered as a collection of single-valued f^ns, each member of which is called a branch of the f^n. It is customary to consider one particular member as a principal branch of the multiple-valued f^n and the value of the f^n corresponding to this branch as the principal value.

If $\omega = z^2$, then to each value of z there corresponds only one value of ω. Hence $\omega = z^2$ is a single-valued f^n of z.

If $\omega^2 = z$, then to each value of z there correspond two values of ω. Hence $\omega^2 = (\sqrt{z})^2$ is a multiple-valued (two-valued) f^n of z.

Branch points and branch cuts (lines)

~~We have~~ Let us consider the f^n $\omega = \sqrt{z}$. Suppose we allow z to make a complete circuit (counter clockwise) around the origin starting from the point A.



We have $z = re^{i\theta}$, $\omega = \sqrt{r}\, e^{\frac{i\theta}{2}}$

so that at A, $\theta = \theta_1$, $r = r_1$ and $\omega = \sqrt{r_1}\, e^{i\theta_1/2}$. After a complete circuit back to A, $\theta = \theta_1 + 2\pi$, $r = r_1$ and $\omega = \sqrt{r_1}\, e^{i\frac{\theta_1+2\pi}{2}} = -\sqrt{r_1}\, e^{i\theta_1/2}$

Thus we do not achieve the same value of ω with which we started. However by making a second complete circuit back to A i.e,

$\theta = \theta_1 + 4\pi$, $r = r_1$, $\omega = \sqrt{r_1}\, e^{i\frac{\theta_1+4\pi}{2}} = \sqrt{r_1}\, e^{i\theta_1/2}$

and we obtain the same value of ω with which we started.

We can describe this fact by stating that if $0 \le \theta < 2\pi$ we are on one branch of the multiple-valued f^n $\omega = \sqrt{z}$, while if $2\pi \le \theta < 4\pi$ we are on the other branch of the f^n.

It is clear that each branch of the f^n is single-valued. In order to keep the f^n single-valued, we set up an artificial barrier such (although any other line from O can be used) which we agree not to cross. This barrier is called <u>branch cut</u> or <u>branch line</u> and the point O is called a <u>branch point</u>.

Let $w = f(z) = \sqrt{z^2+1}$ (a) Show that $z = \pm i$ are branch points of f. (b) Show that a complete circuit around both branch points produces no change in the branches of f (c) Determine branch cuts.

We have $w = \sqrt{z^2+1} = \sqrt{z-i}\,\sqrt{z+i}$. Let $z-i = re^{i\theta}$ and $z+i = te^{i\phi}$. Then $w = \sqrt{rt}\, e^{i\frac{\theta+\phi}{2}}$

Suppose we start with a particular value of z corresponding to $\theta = \alpha_1$, $r = r_1$ and $\phi = \beta_1$, $t = t_1$. Then $w = \sqrt{r_1 t_1}\, e^{i\frac{\alpha_1+\beta_1}{2}}$

As z moves once counterclockwise around i, θ increases to $\alpha_1 + 2\pi$ while ϕ remains the same. Hence $w = \sqrt{r_1 t_1}\, e^{i\frac{\alpha_1+2\pi+\beta_1}{2}} = -\sqrt{r_1 t_1}\, e^{i\frac{\alpha_1+\beta_1}{2}}$ showing that we do not obtain the original value of w, that is, a change of branch occurs. Hence $z = i$ is a branch point.

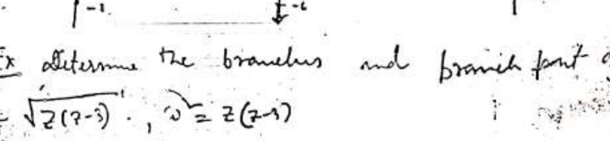
Similarly, if z moves once counterclockwise around $-i$, $w = \sqrt{r_1 t_1}\, e^{i\frac{\alpha_1+\beta_1}{2}}$ becomes

$w = -\sqrt{r_1 t_1}\, e^{i\frac{\alpha_1+\beta_1}{2}}$, that is, a change of branch occurs. Hence $z = -i$ is a branch point. Therefore $z = i$ and $z = -i$ are two branch points.

(b) Let z move along a curve once round i and $-i$ counterclockwise. Then θ increases from α_1 to $\alpha_1 + 2\pi$ and ϕ increases from β_1 to $\beta_1 + 2\pi$. Thus

$$w = \sqrt{r_1 t_1}\, e^{i\frac{\alpha_1+2\pi+\beta_1+2\pi}{2}} = \sqrt{r_1 t_1}\, e^{i\frac{\alpha_1+\beta_1}{2}}$$ and no change in branch is observed.

(c) In order to prevent movement around the branch points we may introduce branch cuts as shown in the following figures (bold lines).



<u>Ex</u> Determine the branches and branch point of $w = f(z) = \sqrt{z(z-3)}$, $w^2 = z(z-3)$

Sol. Clearly ...

which has two branch points namely $z=0$ and $z=3$.
Let $P = ke^{i\alpha}$. Then at P, $w_1 = k^{1/2} e^{i\alpha/2} (ke^{i\alpha} - 3)^{1/2}$
When P makes one complete revolution about the origin we shall $P = ke^{i(\alpha+2\pi)}$. At P, $w_1' = k^{1/2} e^{\frac{i\alpha}{2}} e^{i\pi} (ke^{i(\alpha+2\pi)} - 3)^{1/2}$
$= -k^{1/2} e^{\frac{i\alpha}{2}} (ke^{i\alpha} - 3)^{1/2} = -w_1 = w_2$. Similarly we can see the other branch point is $z=3$. Here P is given by $z-3 = ke^{i\alpha}$. At P $w_1 = \sqrt{(3+ke^{i\alpha})\,ke^{i\alpha}} = k^{1/2} e^{i\alpha/2} \sqrt{(ke^{i\alpha}+3)}$
When P makes one complete circuit about the point 3 we have $P = 3 + ke^{i(\alpha+2\pi)}$. At P,
$$w_1' = \sqrt{(3+ke^{i(\alpha+2\pi)})\,ke^{i(\alpha+2\pi)}} = k^{1/2} e^{i\alpha/2} e^{i\pi} \sqrt{3+ke^{i\alpha}}$$
$$= -k^{1/2} e^{i\alpha/2} \sqrt{3+ke^{i\alpha}} = -w_1 = w_2$$

$|f(z_0)|$ =

$M = |w_0| < |w_1|$