$a_n = 0$ for $n = 1, 2, 3, \ldots$ so that $f(z) = a_0$ i.e., f is a constant. Thus proves the theorem.

Theorem A fn f which is analytic everywhere and has a pole of order m at the point at infinity is a polynomial of degree m.

Proof Since f is analytic for all finite z, by Taylor's theorem we get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

and the series converges for all finite z. So

$f\left(\frac{1}{\zeta}\right) = \sum_{n=0}^{\infty} a_n \frac{1}{\zeta^n}$. Since f has a pole of order m at the point at infinity, $f\left(\frac{1}{\zeta}\right)$ has a pole of order m at the origin. Hence $a_n = 0$ for $n = m+1, m+2, \ldots$ …. Therefore $f\left(\frac{1}{\zeta}\right) = a_0 + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \cdots + \frac{a_m}{\zeta^m}$ and so $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$, which is a polynomial of degree m. This proves the theorem.

Converse of the above theorem

A polynomial of degree m is an integral function and has a polar singularity of order m at infinity.

Proof Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$, where $a_m \neq 0$ be a polynomial of degree m.

Clearly $f(z)$ is a integral function.

Letting $z = \frac{1}{\zeta}$.

$$f\left(\tfrac{1}{\zeta}\right) = a_0 + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \cdots + \frac{a_m}{\zeta^m}.$$

Clearly $\zeta = 0$ is a pole of order m of $f\left(\frac{1}{\zeta}\right)$.

$\therefore z = \infty$ is a pole of order m of $f(z)$.

Residues

Definition If a function f has an isolated at $z = \alpha\ (\neq \infty)$ with Laurent's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n},$$

valid in some deleted neighbourhood $0 < |z-\alpha| < r$, then the coefficient of $\frac{1}{z-\alpha}$ i.e., b_1 in Laurent's expansion is called the residue of f at the singularity α and is usually denoted by $\text{Res}(f; \alpha)$. From the definition we get

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\,dz,$$ where C is any circle given by $C: |z-\alpha| = \rho\ (< r)$.

Clearly the residue of f at the isolated singularity α may also be defined by

$b_1 = \frac{1}{2\pi i} \oint_C f(z)\,dz$, where C is any simple closed contour in the domain $0 < |z-\alpha| < r$, which encloses α and no other singularity of f.

Theorem If f has a pole of order m at $z = \alpha$ then

$$\text{Res}(f;\alpha) = \frac{1}{\lfloor m-1} \lim_{z\to\alpha} \frac{d^{m-1}}{dz^{m-1}} \left[(z-\alpha)^m f(z)\right]$$

Proof Since α is a pole of f of order m, then in some deleted nbd. of α we can write

$$f(z) = \varphi(z) + \frac{b_1}{z-\alpha} + \frac{b_2}{(z-\alpha)^2} + \cdots + \frac{b_m}{(z-\alpha)^m},$$

where φ is analytic in that nbd. and $b_m \neq 0$. So

$$(z-\alpha)^m f(z) = (z-\alpha)^m \varphi(z) + b_1 (z-\alpha)^{m-1} + b_2 (z-\alpha)^{m-2} + \cdots + b_m$$

differentiating $(m-1)$ times we get

$$\frac{d^{m-1}}{dz^{m-1}} \left[(z-\alpha)^m f(z)\right] = \frac{d^{m-1}}{dz^{m-1}} \left[(z-\alpha)^m \varphi(z)\right] + b_1 \lfloor m-1,$$

and so

$$\lim_{z\to\alpha} \frac{d^{m-1}}{dz^{m-1}} \left[(z-\alpha)^m f(z)\right] = \lim_{z\to\alpha} \frac{d^{m-1}}{dz^{m-1}} \left[(z-\alpha)^m \varphi(z)\right] + b_1 \lfloor m-1$$

i.e., $b_1 = \frac{1}{\lfloor m-1} \lim_{z\to\alpha} \frac{d^{m-1}}{dz^{m-1}} \left[(z-\alpha)^m f(z)\right]$,

because $\lim_{z\to\alpha} \frac{d^{m-1}}{dz^{m-1}} \left[(z-\alpha)^m \varphi(z)\right] = 0$.

This proves the theorem.

Particular case If α is a simple pol[e]

$$\text{Res}(f;\alpha) = \lim_{z\to\alpha} (z-\alpha) f(z).$$

Theorem Let $f(z) = \frac{\varphi(z)}{\psi(z)}$, where φ a[nd] analytic at α. If $\varphi(\alpha) \neq 0$ and ψ [has a simple] zero at $z = \alpha$ (i.e., $\psi(\alpha) = 0$, $\psi'(\alpha) \neq 0$) th[en] simple pole of f and $\text{Res}(f;\alpha) =$

Proof Clearly α is a simple pole of f

$$\text{Res}(f;\alpha) = \text{Res}\left(\frac{\varphi}{\psi};\alpha\right) = \lim_{z\to\alpha} (z-\alpha)$$

$$= \lim_{z\to\alpha} \frac{\varphi(z)}{\frac{\psi(z)-\psi(\alpha)}{z-\alpha}} \quad [\because \psi(\alpha) = 0]$$

$$= \frac{\varphi(\alpha)}{\psi'(\alpha)}.$$

Example Find the residue of $\cot z$ at

Soln Since $\cot z = \frac{\cos z}{\sin z}$, we get

$$\text{Res}(\cot z, 0) = \frac{\cos 0}{\sin' 0} = \frac{\cos 0}{\cos 0} = 1$$

Example Find the residues of $f(z) = \frac{z}{?}$ its singularities.

$\therefore$ f has a simple pole at $z=0$ and a pole of order 3 at $z=1$. Now

$$\operatorname{Res}(f;0) = \lim_{z\to 0} z\cdot\frac{z^3-5}{z(z-1)^3} = \lim_{z\to 0}\frac{z^3-5}{(z-1)^3} = -5.$$

$$\operatorname{Res}(f;1) = \frac{1}{\lfloor 3-1}\lim_{z\to 1}\frac{d^{3-1}}{dz^{3-1}}\left[(z-1)^3\cdot\frac{z^3+5}{z(z-1)^3}\right]$$

$$= \frac{1}{\lfloor 2}\lim_{z\to 1}\frac{d^2}{dz^2}\left(\frac{z^3+5}{z}\right)$$

$$= \frac{1}{\lfloor 2}\lim_{z\to 1}\frac{d^2}{dz^2}\left(z^2+\frac{5}{z}\right)$$

$$= \frac{1}{\lfloor 2}\lim_{z\to 1}\frac{d}{dz}\left(2z-\frac{5}{z^2}\right)$$

$$= \frac{1}{2}\lim_{z\to 1}\left(2+\frac{10}{z^3}\right) = \frac{12}{2} = 6.$$

Note This method is recommended in the case of rational functions. For transcendental functions the problem of evaluating the $(m-1)$th derivative at $z=\alpha$ will normally lead to indeterminate forms requiring repeated use of L'Hospital's rules or of a similar device.

Ex Determine the singularities of $\sec\frac{1}{z}$ and find the residues at its poles.

Soln: Here $f(z) = \sec\frac{1}{z} = \dfrac{1}{\cos\frac{1}{z}}$. The singularities of f are precisely the zeros of $\cos\frac{1}{z}$ and the point $z=0$. Now the zeros of $\cos\frac{1}{z}$ are given by $\frac{1}{z} = (2n+1)\pi/2$, $n=0,\pm1,\pm2,\dots$ Since $\frac{d}{dz}\cos\frac{1}{z}\neq 0$ for $z=\frac{1}{(2n+1)\pi/2}$, $n=0,\pm1,\pm2,\dots$, $z=\frac{1}{(2n+1)\pi/2}$ are all simple zeros of $\cos\frac{1}{z}$. So, $z=\frac{1}{(2n+1)\pi/2}$ are all simple poles of $f(z)$.

The point $z=0$ is clearly the limit pt of the poles given by $z=\frac{1}{(2n+1)\pi/2}$, $n=0,\pm1,\pm2,\dots$. Therefore $z=0$ is a nonisolated essential singularity of $f(z)$. Now we find out the residues at the poles. The poles of f occur at $z=\frac{1}{(2n+1)\pi/2}$, $n=0,\pm1,\pm2,\dots$

Therefore, $\operatorname{Res}\left(f;\frac{1}{(2n+1)\pi/2}\right) = \dfrac{[1]_{z=\beta}}{[\cos\frac{1}{z}]'_{z=\beta}}$, with $\beta=\frac{1}{(2n+1)\pi/2}$

$$= \lim_{z\to[(2n+1)\pi/2]^{-1}}\left\{z-\frac{1}{(2n+1)\pi/2}\right\}\cdot\frac{1}{\cos\frac{1}{z}}$$

$$= \lim_{z\to\frac{1}{(2n+1)\pi/2}}\frac{z^2}{\sin\frac{1}{z}} = \frac{1}{\{(2n+1)\pi/2\}^2}\cdot\frac{1}{\sin(2n+1)\pi/2}$$

$$= \frac{(-1)^n}{\{(2n+1)\pi/2\}^2}.$$

Residue at an essential singularity

In this case one usually has to resort to the Laurent's series expansion if it can be found.

Example For $f(z) = e^{-1/z}$ the point $z=0$ is an essential singularity. Also.

$$e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} - \frac{1}{3!}\cdot\frac{1}{z^3} + \cdots$$ from which

we can see that the residue at $z=0$ is the coefficient of $\frac{1}{z}$ and is equal to -1.

Residue at the point at infinity

Let the pt. $z=\infty$ be an isolated singularity of the analytic f^n f. The residue of f at $z=\infty$ is defined as follows:

$$\text{Res}(f;\infty) = \frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\oint_C f(z)\,dz.$$

$$= -\frac{1}{2\pi i}\oint_{C'} f(z)\,dz, \quad \cdots\cdots (1)$$

Where the contour C is an arbitrary closed contour outside of which the f^n f is analytic and does not have any singularity different from infinity.

Cauchy's Residue Theorem

Statement Let f be a single-valued and a[nalytic] within and on a simple closed contour C [except] at a finite number of isolated singular p[oints $z_1, z_2,$] $\ldots, z_n$ within C. Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f; z_k)$$

$= 2\pi i$ (sum of the residues of f [at the] singularities enclosed by C), where the in[tegral] along C is taken in the positive sense.

Proof Let $C_1, C_2, \ldots, C_n$ be n circles with centres $z_1, z_2, \ldots, z_n$ and radii so small that they lie entirely within C and don't overlap. Then f is analytic in the region bo[unded by] C and the circles $c_1, c_2, \ldots, c_n$ so that [by] fundamental theorem

$$\oint_C f(z)\,dz = \sum_{k=1}^{n}\oint_{C_k} f(z)\,dz.$$

Since $\text{Res}(f; z_k) = \frac{1}{2\pi i}\oint_{C_k} f(z)\,dz$, $k = 1, 2, \ldots, n$

it follows that

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f; z_k).$$ This pro[ves the theorem.]

Theorem Let the f^n f be analytic in the e[xtended] plane with the exception of a finite nu[mber]

So, $I = 2\pi i\left(\frac{1}{4}+\frac{1}{4}-\frac{1}{4}-\frac{1}{4}\right)=0$.

(ii) We denote the integral by I and the integrand by f. Then by Cauchy's Residue theorem we get

$I = 2\pi i \, Res(f, 1)$ because the only singularity of f lying within $|z|=2$ is $z=1$ being a double pole.

Now $Res(f,1) = \frac{1}{\lfloor 2-1} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \frac{\sin z}{(z-1)^2(z^2+9)}\right]$

$= \lim_{z\to 1} \frac{d}{dz}\left(\frac{\sin z}{z^2+9}\right)$

$= \lim_{z\to 1} \frac{(z^2+9)\cos z - 2z\sin z}{(z^2+9)^2}$.

$= \frac{10\cos 1 - 2\sin 1}{100} = \frac{5\cos 1 - \sin 1}{50}$

So. $I = 2\pi i \;\frac{5\cos 1 - \sin 1}{50}$.

Definition Let f be an analytic fn and a be a complex number. Then a zero of $f(z)-a$ is called an a-point of f. If $a=\infty$ then a pole of f is called an ∞ point of f.

Theorem Let f be analytic within and on a simple closed contour C except for at most a finite number of poles $\beta_1, \beta_2, \ldots, \beta_n$ within C and let g be regular within and on C and have no singularity there. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the a-points of f within C and f have no a-point on C. Then

$$\frac{1}{2\pi i}\oint_C g(z)\frac{f'(z)}{f(z)-a}\,dz = \sum_{k=1}^{m} p_k\, g(\alpha_k) - \sum_{k=1}^{n} q_k\, g(\beta_k).$$

Where p_k is the order of α_k and q_k is the order of β_k.

Proof Since g is regular at $z=\alpha_k$, we can expand g in a Taylor series about α_k as follows

$g(z) = \sum_{j=0}^{\infty} a_j (z-\alpha_k)^j$ where $a_j = \frac{g^{(j)}(\alpha_k)}{j!}$.

Again since f has an a-pt. of order p_k at α_k, it follows that in some nbd. of α_k

$f(z)-a = (z-\alpha_k)^{p_k}\varphi(z)$, where φ is analytic at α_k and $\varphi(\alpha_k)\neq 0$.

Then $f'(z) = p_k(z-\alpha_k)^{p_k-1}\varphi(z) + (z-\alpha_k)^{p_k}\varphi'(z)$.

Therefore in some nbd. of α_k we get

$$g(z)\times\frac{f'(z)}{f(z)-a} = g(z)\left[\frac{p_k}{z-\alpha_k}+\frac{\varphi'(z)}{\varphi(z)}\right]$$

$$= \frac{a_0 p_k}{z-\alpha_k} + \sum_{j=1}^{\infty} p_k a_j (z-\alpha_k)^{j-1} + g(z)\frac{\varphi'(z)}{\varphi(z)}$$

isolated singular points $z_K (K=1,2,\cdots,N)$ including $z=\infty$ (say $z_N=\infty$). Then $\sum_{K=1}^{N} \text{Res}(f; z_K) = 0$.

Proof Consider the closed contour C containing all $N-1$ singularities z_K, located at a finite distance from the pt. $z=0$. By Cauchy's residue theorem we get

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_{K=1}^{N-1} \text{Res}(f; z_K) \quad \text{------} (1)$$

Also we know that $\text{Res}(f;\infty) = -\frac{1}{2\pi i}\oint_C f(z)\,dz$ ---- (2)

So from (1) and (2) we get

$$\sum_{K=1}^{N-1} \text{Res}(f; z_K) = -\text{Res}(f;\infty) = -\text{Res}(f; z_N)$$

i.e., $\sum_{K=1}^{N} \text{Res}(f; z_K) = 0$ This proves the theorem.

Ex Evaluate $\oint_{|z|=2} \frac{e^z}{z(z-1)^2}\,dz$.

Soln If we denote the integral by I and the integrand by f then by Cauchy's residue theorem we get

$$I = 2\pi i\{\text{Res}(f;0) + \text{Res}(f;1)\},$$ because the only singularities of f lying within $|z|=2$ are (i) only simple pole at $z=0$, (ii) one double pole at $z=1$.

$\therefore\ g(z)\dfrac{f'(z)}{f(z)-a} = \dfrac{p_k\, g(\alpha_k)}{z-\alpha_k} + \psi(z),$

where $\psi(z) = \sum_{j=1}^{\infty} p_k a_j (z-\alpha_k)^{j-1} + g(z)\dfrac{\varphi'(z)}{\varphi(z)}$ is regular at $z=\alpha_k$.

This shows that $g(z)\dfrac{f'(z)}{f(z)-a}$ has a simple pole with residue $p_k\, g(\alpha_k)$ at $z=\alpha_k$ $(k=1,2,\dots,m)$.

Since g is regular at $z=\beta_k$, we can expand g in a Taylor series about β_k as follows

$$g(z)=\sum_{j=0}^{\infty} b_j (z-\beta_k)^j, \text{ where } b_j = \frac{g^{(j)}(\beta_k)}{j!}.$$

Also since f has a pole at β_k of order q_k, we get in some nbd. of β_k

$$f(z)-a = \frac{e(z)}{(z-\beta_k)^{q_k}}, \text{ where } e(z) \text{ is analytic at}$$

β_k and $e(\beta_k)\neq 0$, because β_k is a pole of $f(z)-a$ also.

Then in some nbd. of β_k we obtain

$$g(z)\frac{f'(z)}{f(z)-a} = g(z)\left[\frac{-q_k}{z-\beta_k} + \frac{e'(z)}{e(z)}\right]$$

$$= \frac{-q_k\, g(\beta_k)}{z-\beta_k} - q_k\sum_{j=1}^{\infty} b_j (z-\beta_k)^{j-1} + g(z)\frac{e'(z)}{e(z)}.$$

$$= \frac{-q_k\, g(\beta_k)}{z-\beta_k} + h(z), \text{ where}$$

$h(z) = -q_k\sum_{j=1}^{\infty} b_j (z-\beta_k)^{j-1} + g(z)\dfrac{e'(z)}{e(z)}$ is regular at $z=\beta_k$.

This shows that $g(z)\dfrac{f'(z)}{f(z)-a}$ has a simple pole with residue $-q_k\, g(\beta_k)$ at $z=\beta_k$ $(k=1,2,\dots,n)$.

Since $g(z)\dfrac{f(z)}{f(z)-a}$ has no other singularity, by Cauchy's residue theorem we get

$$\frac{1}{2\pi i}\oint_C g(z)\frac{f'(z)}{f(z)-a}\,dz = \sum_{k=1}^{m} p_k\, g(\alpha_k) - \sum_{k=1}^{n} q_k\, g(\beta_k).$$

This proves the Theorem.

<u>Argument Principle</u>

<u>Statement</u>: Let f be analytic within and on a closed contour C except for atmost a finite number of poles within C and let $f(z)\neq 0$ anywhere on C. Then the excess of the number of zeros over the number of poles of f within C equals $\frac{1}{2\pi}$ times the change in $\arg f(z)$ as z describes the contour C once in the positive sense, a zero or a pole of order m being counted m-times, i.e.,

$$N-P = \frac{1}{2\pi}\left[\arg f(z)\right]_C.$$

<u>Proof</u> Let β_k $(k=1,2,\dots,n)$ be the pole of f with order q_k $(k=1,2,\dots,n)$ within C and α_k be the zero of

f with order p_k $(k=1,2,\ldots,m)$ within C. Then choosing $g(z)=1$ and $a=0$ in the above theorem we get

$$\sum_{k=1}^{m} p_k - \sum_{k=1}^{n} q_k = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz \quad \ldots\ldots (1)$$

If we actually carry out integration, we get

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\left[\log f(z)\right]_C$$

$$\quad // = \frac{1}{2\pi i}\left[\log|f(z)| + i\arg f(z)\right]_C$$

$$= \frac{1}{2\pi}\left[\arg f(z)\right]_C,$$ since $\log|f(z)|$ returns to its original value when we go once round C.

Again if we count the zeros and poles according to their multiplicities, we get

$$N-P = \sum_{k=1}^{m} p_k - \sum_{k=1}^{n} q_k .$$

Therefore from (1) we get

$$N-P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\left[\log f(z)\right]_C$$

$$= \frac{1}{2\pi}\left[\arg f(z)\right]_C .$$ This proves the theorem.

<u>Corollary</u> If f is analytic within and on C and $f(z)\neq 0$ on C then the number of zeros of f within C is given by $N = \frac{1}{2\pi}\left[\arg f(z)\right]_C$, because in this case $P=0$.

Rouche's Theorem

<u>Statement</u> If f and g are analytic within and on a closed contour C and if $|g(z)| < |f(z)|$ for z on C, then the f^n $f(z)$ and $f(z)+g(z)$ have the same number of zeros within C.

<u>Proof</u> We first note that none of f and $f+g$ vanish on C. For, if $f(\alpha)=0$ for some α on C, $|g(\alpha)| < |f(\alpha)| = 0$ which is absurd. Again if $f(\alpha)+g(\alpha)=0$ for some α on C then $f(\alpha) = \ldots$ and so $|f(\alpha)| = |g(\alpha)|$ which contradicts the $|g(z)| < |f(z)|$ on C.

Let N = number of zeros of f within C, [N' = number] of zeros of $f+g$ within C. Therefore by [the argument] principle we get

$$N = \frac{1}{2\pi}\left[\arg f(z)\right]_C \quad \text{and} \quad N' = \frac{1}{2\pi}\left[\arg\{f(z)+g(z)\}\right]_C$$

Now, $$N' = \frac{1}{2\pi}\left[\arg\{f(z)+g(z)\}\right]_C$$

$$= \frac{1}{2\pi}\left[\arg\left\{f(z)\left(1+\frac{g(z)}{f(z)}\right)\right\}\right]_C$$

$$= \frac{1}{2\pi}\left[\arg f(z)\right]_C + \frac{1}{2\pi}\left[\arg\left\{1+\frac{g(z)}{f(z)}\right\}\right]_C$$

$$= N + \frac{1}{2\pi}\left[\arg\left\{1+\frac{g(z)}{f(z)}\right\}\right]_C$$

$$N - N' = \frac{1}{2\pi}\left[\arg\left\{1+\frac{g(z)}{f(z)}\right\}\right]_C \quad \text{-----}(1)$$

[We] now introduce the function

$\omega(z) = 1+\frac{g(z)}{f(z)}$ for z lying within and on C.

$\therefore |\omega(z)-1| = \left|\frac{g(z)}{f(z)}\right| < 1 \;\; \forall z$ on C.

[Th]is shows that as z describes the closed [cur]ve C the corresponding pt. ω describes a [c]urve Γ which lies entirely within the circle $|\omega-1| = 1$. i.e. the curve described by ω lies [en]tirely to the right of the imaginary axis and [h]ence the origin ($\omega=0$) lies outside the curve Γ. [S]o $\arg \omega(z) = \arg\left\{1+\frac{g(z)}{f(z)}\right\}$ returns to its original [v]alue as z describes the closed curve C.

Then $\left[\arg\left\{1+\frac{g(z)}{f(z)}\right\}\right]_C = 0$. So from (1) we get $N' = N$ and the theorem is proved.

Note $\arg z_1 z_2 = \arg z_1 + \arg z_2$

[F]undamental Theorem of Classical Algebra

[The]orem Every polynomial of degree n has n zeros

Proof Let $\phi(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ be a polynomial of degree n so that $a_n \neq 0$. Let $f(z) = a_n z^n$ and $g(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1}$. Then $f(z) + g(z) = \phi(z)$.

Let C denote the circle $|z| = R$ ($R>1$). In the first place f has n zeros within C all the zeros lying at the origin.

Now on C, $|f(z)| = |a_n| R^n$ and

$$|g(z)| = |a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1}|$$
$$\le |a_0| + |a_1| R + |a_2| R^2 + \dots + |a_{n-1}| R^{n-1}$$
$$< R^{n-1}\left[|a_0| + |a_1| + \dots + |a_{n-1}|\right] \quad [\because R>1]$$

Hence $|g(z)| < |f(z)|$ on C

if $R^{n-1}\left[|a_0| + |a_1| + \dots + |a_{n-1}|\right] < |a_n| R^n$,

i.e., if $R > \dfrac{|a_0| + |a_1| + \dots + |a_{n-1}|}{|a_n|}$ ----(1)

Hence by Rouche's theorem $f(z)$ and $f(z)+g(z) = \phi(z)$ will have the same number of zeros within a circle, with centre at the origin and whose radius R satisfies the condition (1).

Hence ϕ has n zeros within such a circle and so such ϕ has also n zeros in the entire complex plane. This completes the proof.

Maximum Modulus Theorem

Statement. Let f be a nonconstant and analytic in a domain D bounded by a closed curve Γ and let f be continuous on $D \cup \Gamma$. If M is the maximum value of $|f(z)|$ in $D \cup \Gamma$, then for every $z \in D$, $|f(z)| < M$.

Proof. Let z_0 be an interior point of D and let $\omega_0 = f(z_0)$. Since the zeros of $f(z) - \omega_0$ are isolated points, we can choose a circle $\gamma : |z - z_0| = \rho$ which lies entirely within D and contains no zero of $f(z) - \omega_0$ other than z_0 within and on it.

Let γ' be the image of γ under $\omega = f(z)$ in the ω-plane. Let $\delta = \frac{1}{2} \inf_{z \in \gamma} |f(z) - \omega_0|$.

Let ω_1 be any point of the open disc $|\omega - \omega_0| < \delta$.

Then for all $z \in \gamma$,

$$|f(z) - \omega_0| \geq 2\delta > |\omega_1 - \omega_0| = |\omega_0 - \omega_1|.$$

So by Rouché's theorem the two functions $f(z) - \omega_0$ and $(f(z) - \omega_0) + (\omega_0 - \omega_1) = f(z) - \omega_1$ have the same number of zeros within γ. So there is at least one point z_1 in $|z - z_1| < \rho$ for which $f(z_1) = \omega_1$.

Let us choose $\omega_1 = \left(|\omega_0| + \frac{\delta}{2}\right) e^{i \arg \omega_0}$.

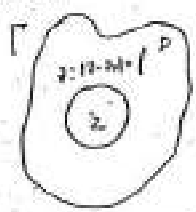
Then $|\omega_1 - \omega_0| = \left| \left(|\omega_0| + \frac{\delta}{2}\right) e^{i \arg \omega_0} - |\omega_0| e^{i \arg \omega_0} \right|$

$= \frac{\delta}{2} < \delta$ and $|\omega_1| = |\omega_0| + \frac{\delta}{2} > |\omega_0|$.

So by the above argument there exists a point z_1 within $\gamma : |z - z_0| = \rho$ for which $f(z_1) = \omega_1$.

Hence $|f(z_1)| = |\omega_1| > |\omega_0| = |f(z_0)|$. Therefore $|f(z_0)|$ cannot be the maximum.

Since z_0 is an arbitrary point of D, it follows that $|f(z)| < M$ for all $z \in D$. This proves the theorem.



Note Since $|f(z)|$ is continuous in the closed region $D \cup \Gamma$, by Maximum Modulus Theorem [the maximum] is attained by $|f(z)|$ on the boundary [Γ].

Minimum Modulus Theorem

Let f be nonconstant and analytic in a domain D bounded by a closed curve Γ and let f be continuous on $D \cup \Gamma$. If $f(z) \neq 0$ in D and m is the minimum value of $|f(z)|$ in $D \cup \Gamma$ then for [every $z \in D$]

m.

If $m = 0$ then the theorem is trivial. So we [...]se that $m > 0$. Let $g(z) = 1/f(z)$. Then by the [...] condition g is analytic in D and continuous [...] $D \cup \Gamma$.

Since $1/m$ is the maximum value of $|g(z)|$ in [...], by maximum modulus theorem we get [...]$(z)| < \frac{1}{m}$ for all $z \in D$. i.e., $|f(z)| > m$ for all [...] in D. This proves the theorem.

Show that if $|a| > e$, the eqn $az^n - e^z = 0$ has [...] roots inside the circle $|z| = 1$.

Let $f(z) = az^n$, $g(z) = -e^z$ so that $f(z) + g(z)$ $= az^n - e^z$. Now on $|z| = 1$

$|f(z)| = |a|\,|z|^n = |a|$ and $|g(z)| = |-e^z| = |e^z| \le e^{|z|} = e$.

Since by hypothesis $e < |a|$, on $|z| = 1$ we get $|g(z)| < |f(z)|$. Hence by Rouche's theorem $f(z) + g(z)$ $= az^n - e^z$ and $f(z) = az^n$ have same number of zeros within $|z| = 1$. Since f has n zeros within $|z| = 1$, all at the origin, the given [eq]uation has n roots within the circle $|z| = 1$.

If $k > 1$, then show that the equation $z^n e^{k-z} = 1$ has n roots within $|z| = 1$.

Soln. The given equation is $z^n e^{k-z} = 1$, i.e., $z^n e^k - e^z = 0$.

Let $f(z) = z^n e^k$ and $g(z) = -e^z$. Then $f(z) + g(z) = z^n e^k - e^z$.

Now on $|z| = 1$,

$|f(z)| = |z^n e^k| = e^k$ and $|g(z)| = |-e^z| = |e^z| \le e^{|z|} = e$.

Since $k > 1$, we get on $|z| = 1$ that $|g(z)| < |f(z)|$. By Rouche's theorem $f(z) + g(z) = z^n e^k - e^z$ and $f(z) = z^n e^k$ have same number of zeros within $|z| = 1$. Since f has n zeros within $|z| = 1$, all at the origin, the given equation has n-roots within the circle $|z| = 1$.

Example If $k > 1$, then show that the equation $z^n e^{k-z} = 1$ has n roots within $|z| = 1$.

Soln. The given eqn is $z^n e^{k-z} = 1$, i.e., $z^n e^k - e^z = 0$.

Let $f(z) = z^n e^k$ and $g(z) = -e^z$. Then $f(z) + g(z)$ $= z^n e^k - e^z$. Now on $|z| = 1$

$|f(z)| = |z^n e^k| = e^k$ and $|g(z)| = |-e^z| = |e^z| \le |e^z| \le e^{|z|} = e$

Since $k > 1$, we get on $|z| = 1$ that $|g(z)| < |f(z)|$. By Rouche's theorem $f(z) + g(z) = z^n e^k - e^z$ and $f(z) = z^n e^k$ have same number of zeros within $|z| = 1$. Since f has n zeros within $|z| = 1$, all at the origin, the given equation has n-roots within the circle $|z| = 1$.

Ex Prove that the roots of the equation $z^5+az+1=0$ lie within $|z|=r$ if $|a|<r^4-\frac{1}{r}$

Soln Let $f(z)=z^5$ and $g(z)=az+1$. Then $f(z)+g(z)$ $=z^5+az+1$. On $|z|=r$, $|f(z)|=|z|^5=r^5$ and

$|g(z)|=|az+1|\le |a|\,|z|+1=|a|r+1$.

So on $|z|=r$, $|g(z)|<|f(z)|$ if $|a|r+1<r^5$

i.e., if $|a|<r^4-\frac{1}{r}$

Then by Rouche's theorem f and $f+g$ have the same number of zeros within $|z|=r$. Since $f(z)=z^5$ has five zeros within $|z|=r$, all at the origin. Hence given equation has five roots within the circle $|z|=r$ if $|a|<r^4-\frac{1}{r}$.

Ex Prove that all the roots of the equation $z^7-5z^3+12=0$ lie in $1<|z|<2$.

Soln Let $f(z)=z^7$ and $g(z)=-5z^3+12$. On $|z|=2$

$|g(z)|\le 5\times 2^3+12=52<128=|f(z)|$. So by Rouche's theorem all the roots of $f(z)+g(z)=0$ lie in $|z|<2$.

Again let $f(z)=12$, $g(z)=z^7-5z^3$. On $|z|=1$,

$|g(z)|\le 1^7+5\times 1^3=6<12=|f(z)|$. Since $f(z)=0$ has no root in $|z|<1$, by Rouche's theorem $f(z)+g(z)=0$ has no root in $|z|<1$. Hence all the roots of the equation $z^7-5z^3+12=0$ lie in $1<|z|<2$.

[If α is a root of $z^7-5z^3+12=0$ with $|\alpha|=1$, then $12=5\alpha^3-\alpha^7$ so that $12=|5\alpha^3-\alpha^7|\le 5|\alpha|^3+|\alpha|^7$ which is impossible. So $z^7-5z^3+12=0$ has no

Conformal Mapping

Rotation of Tangents.

We want to examine the change in dir curves at a point z_0 under a transf $w=f(z)$, where the function $f(z)$ is an at that point.

The derivative of the function at z_0

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}, \quad \ldots \quad (1)$$

exists and is independent of the mann which Δz approaches zero. Thus the value of the complex variable $\frac{\Delta w}{\Delta z}$ approach the absolute value of $f'(z_0)$:

$$\lim_{\Delta z\to 0}\left|\frac{\Delta w}{\Delta z}\right| = |f'(z_0)| \quad \ldots \ldots (2$$

If $f'(z_0)\ne 0$, its argument ψ_0 $(0\le \psi$ the polar representation $f'(z_0)=R_0 e^{i\psi_0}$ has value.

Then it follows from equation (1) that

$$\lim_{\Delta z \to 0} \arg\left(\frac{\Delta w}{\Delta z}\right) = \arg f'(z_0) = \psi_0 \; ; \cdots (3)$$

Since the argument $\frac{\Delta w}{\Delta z}$ also has the range from zero to 2π.

Let C be some curve through z_0 and S be its image under the transformation $w = f(z)$.

If a positive sense of motion along C is prescribed, a corresponding positive sense along S is determined by the function f. When $z_0+\Delta z$ is a point on C in the positive sense from z_0, the limit of the argument of Δz as Δz approaches zero is the angle of inclination of the tangent to C at z_0 drawn in the positive sense. Similarly the argument of Δw approaches the angle of inclination of the tangent to S at w_0, where $w_0 = f(z_0)$. Equation (3) can be written as

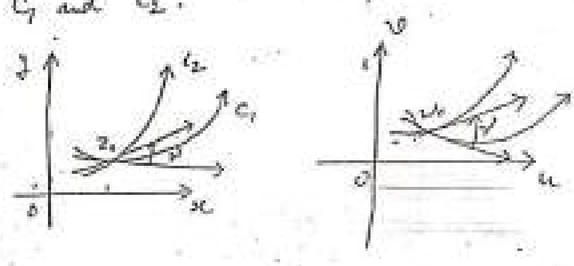
$$\lim_{\Delta z \to 0} \arg \Delta w - \lim_{\Delta z \to 0} \arg \Delta z = \psi_0$$

It follows that if α is the angle of inclination of the curve C at z_0 then the angle of inclination of S at w_0 is $\lim_{\Delta w \to 0} \arg \Delta w = \alpha + \psi_0$.

Thus the directed tangent to a curve C at z_0 is rotated through the angle $\psi_0 = \arg f'(z_0)$, by the transformation $w = f(z)$, provided f is analytic at z_0 and $f'(z_0) \neq 0$.

The angle ψ_0 is the same for all curves through z_0. It is determined by the function f and the point z_0 because $\psi_0 = \arg f'(z_0)$. Consequently any two curves C_1 and C_2 through z_0 are turned through the same angle by the transformation $w = f(z)$.

To state it more precisely, the angle γ at z_0 from C_1 to C_2 is the same both in magnitude and sense, as the angle at w_0 from S_1 to S_2, where the curves S_1 and S_2 are the images of C_1 and C_2.



Definition A mapping or transformation that preserves angles in magnitude and sense between every pair of curves through a point is said to be conformal at the point.

We have, therefore, established the following

Theorem At each point where a function f is analytic and $f'(z) \neq 0$, the mapping $w = f(z)$ is conformal.

Bilinear Transformation

Def. If a, b, c, d are complex constants, then the transformation $w = w(z) = \frac{az+b}{cz+d}$ - - - - (1), where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$, is called a bilinear transformation (briefly B.T.) or a linear fractional transformation or a Möbius transformation.

The expression $ad-bc$ is called the determinant (or jacobian) of the transformation (1).

When $c = 0$, it is simply a linear transformation.

When $c \neq 0$, equation (1) takes the form

$$w = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d}.$$

If $ad - bc = 0$ then w reduces to a constant function and hence the condition $ad-bc \neq$ that the function w is non-constant.

Moreover the transformation (1) is equival[ent to] a succession of transformations:

$z_1 = cz + d$, $z_2 = \frac{1}{z_1}$, $w = \frac{a}{c} + \frac{bc-ad}{c} z_2$

Solving equation (1) for z we get

$$z = \frac{dw - b}{-cw + a} \quad - - - - (2) \quad \text{where } \begin{vmatrix} d & \cdot \\ -c & \cdot \end{vmatrix}$$

The transformation (2) is the inverse of [(1)] which is single valued.

It follows that the inverse of a [B.T. is] another B.T. having the same deter[minant]. From the B.T. (1) and its inverse (2) [it follows] that to every z other than $-\frac{d}{c}$ ($w(z)$ [has a] simple pole at $z = -d/c$) corresponds ju[st one value] of w other than $\frac{a}{c}$ ($z(w)$ has a simple [pole at $w =$] a/c) corresponds just one value of z. [We] suppose that the point at infinity i[n the w-plane] corresponds to the point $z = -\frac{d}{c}$, and [the] point at infinity in the z-plane is m[apped to] the point $w = \frac{a}{c}$. Thus if $c \neq 0$, $z = \infty$ [corresponds] to $w = \frac{a}{c}$ and $z = -d/c = -d/c$ corresponds now follows that the B.T. (1) sets up a 1-1 [co]rrespondence between the points of the extended [z-]plane and the points on the extended ω-plane. [wh]en $c=0$, $z=\infty$ corresponds to $\omega=\infty$.

[Pro]perties of Bilinear Transformations.

[I] B.T. are conformal mappings of the extended z-plane [on]to the extended ω-plane.

[Pf] Let $\omega = \frac{az+b}{cz+d}$ $(ad-bc\neq 0)$ be a B.T.

Then $\frac{d\omega}{dz} = \frac{a(cz+d)-c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0$

[a]nd so $\omega(z)$ is a conformal mapping.

[II] The product (composition) of two successive B.T. [is] again a B.T. $\quad \zeta = f(z)$, $\omega = g(\zeta) = g_\circ f(z)$

[Pf] Let $\zeta = \frac{a_1 z+b_1}{c_1 z+d_1}$, ... (1) $\quad a_1d_1 - b_1c_1 \neq 0$

[a]nd $\omega = \frac{a_2\zeta+b_2}{c_2\zeta+d_2}$, ... (2) $\quad a_2d_2 - b_2c_2 \neq 0$ be [t]wo B.T.

[Su]bstituting the expression for ζ into the formula [for] ω we find $\omega = \frac{az+b}{cz+d}$ ... (3),

[wher]e $a = a_1a_2 + b_2c_1$, $b = a_2b_1 + b_2d_1$, $c = a_1c_2 + d_2c_1$, $d = b_1c_2 + d_1d_2$.

Now $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a_1a_2 + b_2c_1 & a_2b_1 + b_2d_1 \\ a_1c_2 + c_1d_2 & b_1c_2 + d_1d_2 \end{vmatrix}$

$= \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \cdot \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} \neq 0.$

and so (3) is a B.T.. Thus the composition of two B.T. is again a B.T.

III The inverse of a B.T. is also a B.T.

This assertion has already been proved.

IV The identity mapping $\omega = z$ is trivially a B.T.

V The associative law for composition of B.T. holds.

Proof Let $T_1 : \zeta = \frac{a_1 z+b_1}{c_1 z+d_1}$ ... (1) $\quad a_1d_1 - b_1c_1 \neq 0$.

$T_2 : \omega = \frac{a_2\zeta+b_2}{c_2\zeta+d_2}$ ... (2) $\quad a_2d_2 - b_2c_2 \neq 0$

$T_3 : \lambda = \frac{a_3\omega+b_3}{c_3\omega+d_3}$ ... (3) $\quad a_3d_3 - b_3c_3 \neq 0$.

Now. $T_3T_2 : \lambda = \frac{a_3\frac{a_2\zeta+b_2}{c_2\zeta+d_2} + b_3}{c_3\frac{a_2\zeta+b_2}{c_2\zeta+d_2} - d_3}$ $\quad z \xrightarrow{T_1} \zeta \xrightarrow{T_2} \omega \xrightarrow{T_3} \lambda$ $\quad T_3 \circ T_2 \circ T_1(z)$

$= \frac{(a_2a_3 + b_3c_2)\zeta + (a_3b_2 + b_3d_2)}{(a_2c_3 + c_2d_3)\zeta + (b_2c_3 + d_2d_3)}$

$(T_3T_2)T_1 : \lambda = \dfrac{(a_1a_2a_3 + c_1b_3c_2 + a_3b_2c_1 + b_3c_1d_2)z + (a_1a_3b_1 + b_1b_3c_1 + a_3b_2d_1 + b_3d_1d_2)}{(a_1c_2c_3 + c_1c_2d_3 + b_2c_1c_3 + c_1d_1d_3)z + (c_2b_1c_3 + b_1c_2d_3 + b_2c_3d_1 + d_1d_2d_3)}$

$= \dfrac{Az + B}{Cz + D}$, say.

Similarly we can show that $T_3(T_2T_1) : \lambda = \dfrac{Az+B}{Cz+D}$

and so associative law for composition of B.T. holds.

Hence we can state

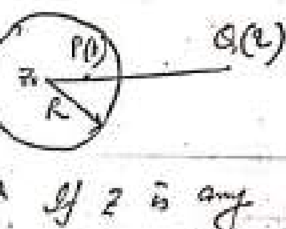
Theorem The set of all B.T. form a group.

$\left|\dfrac{Re^{i\theta} - \rho e^{i\lambda}}{Re^{i\theta} - \frac{R^2}{\rho}e^{i\lambda}}\right|$

Inverse Point

Let C be a ~~point~~ circle of radius R with centre at z_0. Two points p and q are said to be inverse w.r.t. the circle C if they are collinear with the centre and lie on the same side of it and if the product of their distances from the centre is equal to R^2.

Clearly q is exterior to C iff p is interior to C. If q is on C then q coincides with p.



Note $z = 0$ and $z = \infty$ are also considered as a pair of inverse points.

Thus if $p = z_0 + \rho e^{i\lambda}$ then $q = z_0 + \frac{R^2}{\rho}e^{i\lambda}$. If z is any ... then $z = z_0 + Re^{i\theta}$. Therefore

$\left|\dfrac{z-p}{z-q}\right| = \dfrac{\rho}{R}$. That is, therefore, a ...

of equation of a circle. Conversely, it can ... that $\left|\dfrac{z-p}{z-q}\right| = k\ (\neq 1)$ represents a circle ... z-plane w.r.t. which p and q are inverse ...

In the particular case when $k = 1$, z is equi... from the points p and q and hence li... perpendicular bisector of the line joining them.

Theorem A B.T. transforms a circle into ... and inverse points into inverse points.

In the particular case in which the ... a st. line, inverse points becomes points ... about the line.

Proof Let $\left|\dfrac{z-p}{z-q}\right| = k$ be a circle (or a st. ...) p and q as inverse points (or symmetric p...) ... (i)

Let $w = \dfrac{az+b}{cz+d}$, ad $- bc \neq 0$, ...

be a B.T. Solving for z we get $z = \dfrac{dw - b}{-cw + a}$

Then the circle is transformed into

$\left|\dfrac{dw - b - p(-cw + a)}{dw - b - q(-cw + a)}\right| = k$

i.e., $\left|\dfrac{w(cp + d) - (ap + b)}{w(cq + d) - (aq + b)}\right| = k$

$z - p = \rho e^{i\lambda}$ ...
$z - q = Re^{i\theta}$ ...
$= \frac{R}{\rho}$ ...
$\left|\frac{z-p}{z-q}\right| = \frac{\rho}{R}$ ...
$= \frac{\rho}{R}$

i.e., $\left| \dfrac{\omega - \frac{ap+b}{cp+d}}{\omega - \frac{aq+b}{cq+d}} \right| = k \left| \dfrac{cq+d}{cp+d} \right|$

i.e., $\left| \dfrac{\omega-\alpha}{\omega-\beta} \right| = k' \quad \ldots (2)$

where $\alpha = \dfrac{ap+b}{cp+d}$, $\beta = \dfrac{aq+b}{cq+d}$, $k' = k\left|\dfrac{cq+d}{cp+d}\right|$.

Thus the map of the circle $\left|\dfrac{z-p}{z-q}\right| = k$, under the B.T. given by (1) is a circle (or a st. line) $\left|\dfrac{\omega-\alpha}{\omega-\beta}\right| = k'$ with respect to which α and β are inverse points (or symmetric points) which are respectively the maps of p and q. This proves the theorem.

Find the B.T. which maps the points $1, i, -1$ in the z-plane into the points $0, 1, \infty$ in the ω-plane. Show that by means of this fn. the area of the circle $|z|=1$ is represented in the ω-plane by the half-plane above the real axis.

Soln. Let $\omega = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$ be the required B.T. Since

$1 \to 0$, $0 = \dfrac{a+b}{c+d}$ i.e., $a+b=0 \quad \ldots (1)$

$i \to 1$, $1 = \dfrac{ai+b}{ci+d} \quad \ldots (2)$

$-1 \to \infty$, $\infty = \dfrac{-a+b}{-c+d}$ i.e., $-c+d=0 \quad \ldots (3)$

From (1), (2) and (3) we get

$\omega = \dfrac{az-a}{cz+c} = \dfrac{a}{c}\cdot\dfrac{z-1}{z+1} = \dfrac{i+1}{i-1}\cdot\dfrac{z-1}{z+1}$ (by (2))

$= \dfrac{1}{i}\,\dfrac{z-1}{z+1}$, which is the required B.T.

Let z be any pt. in the z-plane. Its image is given by

$\omega = \dfrac{1}{i}\,\dfrac{re^{i\theta}-1}{re^{i\theta}+1} = i\,\dfrac{(1-r^2) - 2ir\sin\theta}{(1+r\cos\theta)^2 + r^2\sin^2\theta}$, putting $z = re^{i\theta}$.

If $\omega = u+iv$, then

$u = \dfrac{2r\sin\theta}{(r\cos\theta+1)^2 + r^2\sin^2\theta}$, $v = \dfrac{1-r^2}{(r\cos\theta+1)^2+r^2\sin^2\theta}$

Case I If $r<1$, $\operatorname{Im}(\omega) > 0$ i.e., the image of a point lying inside the unit circle $|z|=1$ lies in the upper half plane.

Case II If $r>1$, $\operatorname{Im}(\omega) < 0$, i.e., the map of a point outside $|z|=1$ lies in the lower half plane i.e., the function maps the domain $|z|>1$ onto the lower half plane.

Case III The map of a point lying on $|z|=1$ lie on the real axis in the ω-plane, because then $r=1$

and so $Im(w)=0$.

Thus combining the above cases we conclude that the area of the circle $|z|=1$ in the z-plane is represented in the w-plane by the half plane above the real axis.

Ex Show that the tran^{n} $w=\frac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x=0$ onto the line $4u+3=0$.

Soln Given tran^{n} is clearly a B.T. The inverse tran^{n} is given by

$$z=\frac{4w+3}{w-2} \quad \text{.......} (1)$$

Now the equation of the circle can be written as

$$|z|^2-4Re\,z=0.$$

i.e, $|z|^2-2(z+\bar{z})=0$

i.e, $z\bar{z}-2(z+\bar{z})=0 \quad \text{.....} (2)$

Substituting for z and $\bar{z}$ from (1) in (2) we get

$$\frac{4w+3}{w-2}\cdot\frac{4\bar{w}+3}{\bar{w}-2}-2\left(\frac{4w+3}{w-2}+\frac{4\bar{w}+3}{\bar{w}-2}\right)=0$$

i.e, $(4w+3)(4\bar{w}+3)-2\left[(4w+3)(\bar{w}-2)+(4\bar{w}+3)(w-2)\right]=0$

i.e, $(4w+3)(4\bar{w}+3-2\bar{w}+4)-2(4\bar{w}+3)(w-2)=0$

i.e, $(4w+3)(2\bar{w}+7)-2(4\bar{w}+3)(w-2)=0$.

i.e, $8w\bar{w}+6\bar{w}+28w+21-8w\bar{w}-6w+16\bar{w}$ ...

i.e, $22(w+\bar{w})+33=0$

i.e, $2(w+\bar{w})+3=0$

i.e, $4u+3=0$ $[\because w+\bar{w}=2Re(w)=2u]$,

the required line.

Ex Show that $\left|\frac{z-p}{z-q}\right|=k(\neq 1)$ represents a ... to which p and q are inverse points.

Soln From the given eq^{n} we get

$$|z-p|^2=k^2|z-q|^2$$

i.e, $(z-p)(\bar{z}-\bar{p})=k^2(z-q)(\bar{z}-\bar{q})$

i.e, $z\bar{z}+p\bar{p}-(z\bar{p}+\bar{z}p)=k^2\{z\bar{z}+q\bar{q}-(z\bar{q}+\bar{z}q)\}$

i.e, $|z|^2-2Re(\bar{p}z)+|p|^2=k^2\{|z|^2-2Re(\bar{q}z)$ ...

i.e, $(1-k^2)|z|^2-2Re\{\bar{p}z-k^2\bar{q}z\}+|p|^2-k^2|q|^2$ ...

i.e, $|z|^2-2\frac{Re\{(\bar{p}-k^2\bar{q})z\}}{1-k^2}+\frac{|p|^2-k^2|q|^2}{1-k^2}=0$

i.e, $\left|z-\frac{p-k^2q}{1-k^2}\right|^2=\frac{|p-k^2q|^2}{(1-k^2)^2}-\frac{|p|^2-k^2|q|^2}{1-k^2}$

$$=\frac{|p-k^2q|^2}{(1-k^2)^2}-\frac{|p|^2-k^2|q|^2}{1-k^2}$$

$$=\frac{|p-k^2q|^2-(1-k^2)\{|p|^2-k^2|q|^2\}}{(1-k^2)^2}=k^2 \ldots$$

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we obtain

$$\left|z - \frac{p-k^2q}{1-k^2}\right| = \frac{k|p-q|}{|1-k^2|}$$

The equation therefore represents a circle with centre $z_0 = \frac{p-k^2q}{1-k^2}$ and radius $R = \frac{k|p-q|}{|1-k^2|}$.

Also,

$$p - z_0 = p - \frac{p-k^2q}{1-k^2} = \frac{k^2(q-p)}{1-k^2} \text{ and}$$

$$q - z_0 = q - \frac{p-k^2q}{1-k^2} = \frac{q-p}{1-k^2}, \text{ so that}$$

$\frac{p-z_0}{q-z_0} = k^2$ is real and positive and

$$|p-z_0||q-z_0| = \frac{k^2|p-q|}{|1-k^2|} \cdot \frac{|p-q|}{|1-k^2|} = \frac{k^2|p-q|^2}{|1-k^2|^2} = R^2$$

Hence p and q are inverse points. In the particular case $k=1$, z is equidistant from the points p and q, and therefore lies on the perpendicular bisector of the line joining them.

Show that the B.T. which carries the points $z = i, 0, -i$ respectively into $w = 0, -1$, and ∞ maps.
(1) the real axis $Im(z)=0$ on $|w|=1$,
(2) the upper half plane $Im(z)>0$ on $|w|<1$,
(3) the lower half plane on $|w|>1$.

Sol: $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ be the required B.T. Since

$i \to 0$, $0 = \frac{ai+b}{ci+d}$ i.e, $ai+b=0$ ....(1)

$0 \to -1$, $-1 = \frac{b}{d}$ i.e, $b=-d$ .....(2)

$-i \to \infty$, $\infty = \frac{-ai+b}{-ci+d}$ i.e., $-ci+d=0$ ...(3).

From (1), (2) and (3) we get

$$w = \frac{az-ai}{cz+ci} = \frac{a}{c} \cdot \frac{z-i}{z+i} = \frac{z-i}{z+i} \quad \left[\because \frac{a}{c} = \frac{ai}{ci} = -\frac{b}{d} = \frac{d}{d} = 1\right]$$

which is the required B.T.

(1) any point on the real axis $Im(z)=0$ can be taken as $z=x$. Then its image is $w = \frac{x-i}{x+i}$ so that $|w| = \left|\frac{x-i}{x+i}\right| = \sqrt{\frac{1+x^2}{1+x^2}} = 1$.

(2) any point on the upper half plane $Im(z)>0$ can be taken as $z=x+iy$, $y>0$. Then its image is $w = \frac{x+i(y-1)}{x+i(y+1)}$, so that

$$|w| = \left|\frac{x+i(y-1)}{x+i(y+1)}\right| = \sqrt{\frac{x^2+(y-1)^2}{x^2+(y+1)^2}} < 1 \quad [\because y>0]$$

So, the map of the upper half plane $Im(z)>0$ is the region $|w|<1$.

(3) any point on the lower half plane $\text{Im}(z)<0$ can be taken as $z=x+iy$, $y<0$. Then its image is $w=\frac{x+i(y-1)}{x+i(y+1)}$ so that

$$|w| = \left|\frac{x+i(y-1)}{x+i(y+1)}\right| = \sqrt{\frac{x^2+(y-1)^2}{x^2+(y+1)^2}} > 1 \quad [\because y<0]$$

So the map of the lower half plane $\text{Im}(z)<0$ is the region $|w|>1$.

Ex Find all the B.T. of the unit circle $|z|\le 1$ into the unit circle $|w|\le 1$.

Soln Let $w=\frac{az+b}{cz+d}$, Here $w=0, \infty$ must correspond to inverse points $z=\alpha$, $z=\frac{1}{\bar\alpha}$, where $|\alpha|<1$.

Hence $-\frac{b}{a}=\alpha$ and $-\frac{d}{c}=\frac{1}{\bar\alpha}$.

So. $w=\frac{a(z+\frac{b}{a})}{c(z+\frac{d}{c})} = \frac{a}{c}\frac{z-\alpha}{z-\frac{1}{\bar\alpha}} = \frac{a\bar\alpha}{c}\;\frac{z-\alpha}{\bar\alpha z-1}$.

The point $z=1$ corresponds to a point on $|w|=1$.

Hence $\left|\frac{a\bar\alpha}{c}\cdot\frac{1-\alpha}{\bar\alpha-1}\right| = \left|\frac{a\bar\alpha}{c}\right| = 1$

So we put $\frac{a\bar\alpha}{c}=e^{i\lambda}$, λ is real.

Hence $w=e^{i\lambda}\frac{z-\alpha}{\bar\alpha z-1}$, where λ is real.

This is the required result. For ($b<1$) then

$$|w| = \left|\frac{e^{i\theta}-be^{i\beta}}{be^{i(\theta-\beta)}-1}\right| = \left|\frac{e^{i\theta}-be^{i\beta}}{be^{i\theta}-e^{i\beta}}\right|$$

$$= \left|\frac{(\cos\theta-b\cos\beta)+i(\sin\theta-b\sin\beta)}{(b\cos\theta-\cos\beta)+i(b\sin\theta-\sin\beta)}\right|$$

$$= \frac{\sqrt{(\cos\theta-b\cos\beta)^2+(\sin\theta-b\sin\beta)^2}}{\sqrt{(b\cos\theta-\cos\beta)^2+(b\sin\theta-\sin\beta)^2}}$$

$$= \frac{\sqrt{1+b^2-2b\cos(\theta-\beta)}}{\sqrt{1+b^2-2b\cos(\theta-\beta)}} = 1$$

If $z=re^{i\theta}$ where $r<1$, then

$$|z-\alpha|^2-|\bar\alpha z-1|^2 = (re^{i\theta}-be^{i\beta})(re^{-i\theta} \ldots$$

$$-(rbe^{i(\theta-\beta)}-1)(rbe^{-i(\theta \ldots$$

$$=\{r^2+b^2-2rb\cos(\theta-\beta)\}-\{b^2r^2+1-\ldots$$

$$=r^2+b^2-b^2r^2-1 = (r^2-1)(1-b^2)<0$$

$r<1$ and $b<1$

So $|z-\alpha|^2<|\bar\alpha z-1|^2$ i.e., $|z-\alpha| \ldots$

$|w|<1$.

If we are also given that $z= \ldots$ $w=0$, then $\alpha=0$ and the ...

$w = e^{i\lambda} z$.

Find all the b.t. of the half plane $\operatorname{Im}(z) \geq 0$ to the circle $|w| \leq 1$.

Soln. Two points $z, \bar{z}$ symmetrical about the real axis in the z-plane correspond to the points $w, \frac{1}{\bar{w}}$ inverse w.r.t. the unit circle in the w-plane. In particular, the origin and the pt. at infinity are inverse pts. w.r.t. the unit circle. Since conjugate values of z in the z-plane are inverse points w.r.t. the real axis, it follows that the origin and the pt. at infinity in the w-plane correspond to conjugate values of z.

Let $w = \frac{az+b}{cz+d}$ be the required transformation.

Now $w=0$, $w=\infty$ correspond to $z = -\frac{b}{a}, -\frac{d}{c}$.

Hence we may write $\alpha = -\frac{b}{a}$, $\bar{\alpha} = -\frac{d}{c}$.

So, $w = \frac{a}{c} \frac{z+\frac{b}{a}}{z+\frac{d}{c}} = \frac{a}{c} \frac{z-\alpha}{z-\bar{\alpha}}$.

The point $z=0$ must correspond to a point on $|w|=1$.

So $\left| \frac{a}{c} \frac{-\alpha}{-\bar{\alpha}} \right| = 1$ i.e., $\left|\frac{a}{c}\right| = 1$

If we put $\frac{a}{c} = e^{i\lambda}$, where λ is real, and we get

$w = e^{i\lambda} \frac{z-\alpha}{z-\bar{\alpha}}$ - - - - (1)

Since $z = \alpha$ gives $w=0$, α must be a pt of the upper half plane i.e., $\operatorname{Im}(\alpha) > 0$. With this condition the f: (1) gives the required representation. For, if z is real, we get $|w|=1$ and if $\operatorname{Im}(z) > 0$ then $|z-\alpha| < |z-\bar{\alpha}|$ and so $|w| < 1$.

Exercise Find all the B.T. of the half plane $\operatorname{Re}(z) \geq 0$ into the unit circle $|w| \leq 1$.

Hints Take $-\frac{b}{a} = \alpha$, $-\frac{d}{c} = -\bar{\alpha}$ and proceed as above. $\left[w = e^{i\lambda} \frac{z-\alpha}{z+\bar{\alpha}} \; (\operatorname{Re}(\alpha) > 0) \text{ and } \lambda \text{ is real} \right]$

Ex Find all the B.T. of the unit circle $|z| \leq 1$ into the half plane $\operatorname{Im}(w) \geq 0$.

Soln Two points $z, \frac{1}{\bar{z}}$ inverse w.r.t. the unit circle $|z|=1$ correspond to the points $w, \bar{w}$ symmetrical about the real axis in the w-plane. Since 0 and ∞ are inverse points w.r.t. the unit circle $|z|=1$, they are mapped to conjugate points in the w-plane.

Let $w = \frac{a+bz}{c+dz}$ be the required B.T.

Since 0 and ∞ are mapped onto $\frac{a}{c}$ and $\frac{b}{d}$ respectively, we can write $\alpha = \frac{a}{c}$ and $\bar{\alpha} = \frac{b}{d}$.

So, $w = \frac{\alpha c + \bar{\alpha} d z}{c+dz} = \frac{\bar{\alpha} z + \alpha \frac{c}{d}}{\ldots}$

The point $\omega=0$ must be the image of a point on $|z|=1$. So, $z=-\frac{\alpha}{\bar{\alpha}}\cdot\frac{c}{\lambda}$ lies on $|z|=1$ and hence $\left|-\frac{\alpha}{\bar{\alpha}}\cdot\frac{c}{\lambda}\right|=1$ i.e. $\left|\frac{c}{\lambda}\right|=1$

Hence we put $\frac{c}{\lambda}=e^{i\lambda}$, where λ is real and we get

$$\omega=\frac{\bar{\alpha}z+\alpha e^{i\lambda}}{z+e^{i\lambda}}$$

Since $z=0$ gives $\omega=\alpha$, α must be a point in the upper half plane and so $Im(\alpha)>0$. For $z=re^{i\theta}$ and $\alpha=ae^{ib}$, where $a>0$, $\sin b>0$ we see that

$$Im(\omega)=\frac{a(1-r^2)\sin b}{r^2(\cos\theta+\cos\lambda)+(r\sin\theta+\sin\lambda)^2}$$

Case I If $r<1$, $Im(\omega)>0$ i.e. the image of a point lying inside the unit circle $|z|=1$ lies in the upper half plane.

Case II If $r>1$, $Im(\omega)<0$, i.e. the image of a point outside $|z|=1$ lies in the lower half plane i.e. the B.T. maps the domain $|z|>1$ onto the lower half plane.

Case III The map of a point lying on $|z|$ ... the real axis in the ω-plane because and so $Im(\omega)=0$

Thus combining the above cases we co... the area of the circle $|z|=1$ in the z-... represented in the ω-plane by the half p... the real axis.

Ex Find all the B.T. of the unit circ... the half plane $Re(\omega)\geq 0$.

Hints $\alpha=\frac{a}{c}$, $\bar{\alpha}=-\frac{b}{\lambda}$ and proceed as ab...

$$\omega=\frac{-\bar{\alpha}z+\alpha e^{i\lambda}}{z+e^{i\lambda}},\ \text{where } \lambda \text{ is real}$$

Definition Cross ratio of four points z_1, z_2, z_3... in this order is defined as

$$(z_1,z_2,z_3,z_4)=\frac{z_1-z_2}{z_2-z_3}\cdot\frac{z_3-z_4}{z_4-z_1}$$

Ex If for any B.T. $\omega_1,\omega_2,\omega_3,\omega_4$ are the ... corresponding z_1,z_2,z_3,z_4 respectively show that

$$(\omega_1,\omega_2,\omega_3,\omega_4)=(z_1,z_2,z_3,z_4).$$

In other words, show that the cross ratio is ... by any B.T.

Let $T: w = \frac{a+bz}{c+dz}$, $ad-bc \neq 0$ be a B.T.

$w_i = \frac{a+bz_i}{c+dz_i}$, $i = 1,2,3,4$.

$w_2 - w_1 = \frac{(ad-bc)(z_1-z_2)}{(c+dz_2)(c+dz_1)}$,

$w_4 - w_3 = \frac{(ad-bc)(z_3-z_4)}{(c+dz_3)(c+dz_4)}$,

$w_4 - w_1 = \frac{(ad-bc)(z_4-z_1)}{(c+dz_4)(c+dz_1)}$ and

$w_2 - w_3 = \frac{(ad-bc)(z_3-z_2)}{(c+dz_2)(c+dz_3)}$

$$(w_1, w_2, w_3, w_4) = \frac{w_1-w_2}{w_2-w_3} \cdot \frac{w_3-w_4}{w_4-w_1}$$

$$= \frac{w_2-w_1}{w_2-w_3} \cdot \frac{w_4-w_3}{w_4-w_1}$$

$$= \frac{\frac{(ad-bc)(z_1-z_2)}{(c+dz_2)(c+dz_1)}}{\frac{(ad-bc)(z_3-z_2)}{(c+dz_2)(c+dz_3)}} \cdot \frac{\frac{(ad-bc)(z_3-z_4)}{(c+dz_3)(c+dz_4)}}{\frac{(ad-bc)(z_1-z_4)}{(c+dz_1)(c+dz_4)}}$$

$$\frac{z_1-z_2}{z_2-z_3} \cdot \frac{z_3-z_4}{z_4-z_1} = (z_1, z_2, z_3, z_4)$$

Ex Show that every B.T. with two noninfinite fixed points α, β can be expressed as.

$$\frac{w-\alpha}{w-\beta} = k\frac{z-\alpha}{z-\beta}, \text{ Where } k \text{ is a constant satisfying}$$

$$k + \frac{1}{k} = \frac{b^2+c^2+2ad}{bc-ad}, \; w = \frac{a+bz}{c+dz} \;\Big|\; w = \frac{az+b}{cz+d}, \; k+\frac{1}{k} = \frac{a^2+d^2+2bc}{ad-bc}$$

Soln: Suppose that the B.T. with α, β as fixed points maps any point $z = \gamma$ onto $w = \delta$. Then α, β, γ, z are transformed to α, β, δ, w respectively. So by the property of cross ratio preservation, we get

$$(\alpha, \gamma, \beta, z) = (\alpha, \delta, \beta, w)$$

i.e., $\frac{\alpha-\gamma}{\gamma-\beta} \cdot \frac{\beta-z}{z-\alpha} = \frac{\alpha-\delta}{\delta-\beta} \cdot \frac{\beta-w}{w-\alpha}$

i.e., $\frac{\gamma-\alpha}{\gamma-\beta} \cdot \frac{z-\beta}{z-\alpha} = \frac{\delta-\alpha}{\delta-\beta} \cdot \frac{w-\beta}{w-\alpha}$

i.e., $\frac{w-\alpha}{w-\beta} = k\frac{z-\alpha}{z-\beta}$, ---- (1)

where $k = \frac{\delta-\alpha}{\delta-\beta} \cdot \frac{\gamma-\beta}{\gamma-\alpha}$.

Since α, β, are the fixed points of the trans $w = \frac{az+b}{cz+d}$

$w = \frac{a+bz}{c+dz}$ ---- (2)

we see that α, β are the roots of the equation.

$dz^2 + (c-b)z - a = 0$

So $\alpha + \beta = \frac{b-c}{d}$, and $\alpha\beta = -\frac{a}{d}$.

Since $z = \infty$ corresponds to $w = \frac{b}{d}$ in (2) we get from (1) by putting $z = \infty$, $w = \frac{b}{d}$

$$\frac{\frac{b}{d} - \alpha}{\frac{b}{d} - \beta} = K \quad \text{i.e., } K = \frac{b - \alpha d}{b - \beta d} = \frac{c + \beta d}{c + \alpha d}$$

$$\text{So } k + \frac{1}{k} = \frac{b - \alpha d}{b - \beta d} + \frac{b - \beta d}{b - \alpha d}$$

$$= \frac{(b-\alpha d)^2 + (b - \beta d)^2}{(b - \beta d)(b - \alpha d)}$$

$$= \frac{b^2 + \alpha^2 d^2 - 2\alpha b d + b^2 + \beta^2 d^2 - 2\beta b d}{b^2 - \alpha b d - \beta b d + \alpha\beta d^2}$$

$$= \frac{2b^2 - 2bd(\alpha+\beta) + d^2\{(\alpha+\beta)^2 - 2\alpha\beta\}}{b^2 - bd(\alpha+\beta) + d^2\alpha\beta}$$

$$= \frac{2b^2 - 2bd\frac{b-c}{d} + d^2\{(\frac{b-c}{d})^2 + \frac{2a}{d}\}}{b^2 - bd\frac{b-c}{d} - d^2\frac{a}{d}}$$

$$= \frac{b^2 + c^2 + 2ad}{bc - ad}$$. Thus k is a root of

$(bc - ad)t^2 - (b^2 + c^2 + 2ad)t + bc - ad = 0$.

$k \neq 0, 1 \ldots (\sqrt{k} + \frac{1}{\sqrt{k}})^2 = \frac{(b+c)^2}{bc - ad}$.

Fixed Point of a B.T.

Consider any B.T. $w = \frac{a + bz}{c + dz}$, $ad - bc \neq$ [illegible] that w and z are represented by the p[illegible] same plane. Any point z which is f[illegible] which coincides with its transform satisf[illegible]

$z = \frac{a + bz}{c + dz}$ i.e., $dz^2 + (c-b)z - a = 0$ ... [illegible]

So a fixed point is given by the root of [illegible] $d \neq 0$. Then the eqn (1) has the roots $\frac{b - c \pm \sqrt{(b-c)^2 + 4ad}}{2d}$. So we have two or on[illegible] fixed point according as $(b-c)^2 + 4ad \neq 0$ [illegible]

Suppose $d = 0$, so $c \neq 0$. In this case the [illegible] $w = \frac{a}{c} + \frac{b}{c}z$. It is easily seen that ∞ is [illegible] point another fixed point z is given by $z = \frac{a}{c}$ [illegible] or, $z(c-b) = a$ i.e., $z = \frac{a}{c-b}$. So if $c - b \neq 0$, th[illegible] two fixed points namely ∞ and $\frac{a}{c-b}$. If $c-b$ [illegible] only fixed point ∞. We therefore obtain the [illegible] possibilities of fixed point of the B.T.

$w = \frac{a + bz}{c + dz}$, $ad - bc \neq 0$.

(1) $d \neq 0$, $(b-c)^2 + 4ad \neq 0$, two noninfinite fix[illegible]

(2) $d \neq 0$, $(b-c)^2 + 4ad = 0$, one noninfinite fixe[illegible]

(3) $d = 0$, $c - b \neq 0$ one infinite and other [illegible]

(4) $d = 0$, $c - b = 0$ " " fixed poi[illegible]

Contour Integration

Evaluation of real definite integrals.

A variety of real definite integrals can be evaluated with the help of Cauchy's residue theorem. In each case we must choose a suitable contour along which integration is to be performed and the process is known as contour integration.

We shall consider in this section example of some basic types which lend themselves to this technique.

(1) Integration around the unit circle

An integral of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$, where the integrand is a rational function of $\sin\theta$ and $\cos\theta$, can be evaluated by writing $e^{i\theta} = z$. Then $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$. The integral takes the form $\int_C f(z)\,dz$, where f is a rational function of z and C is the unit circle $|z| = 1$. Hence the integral is equal to $2\pi i$ times the sum of residues at those poles of f which are within C.

Show that $\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,d\theta = \frac{2\pi a^2}{1-a^2}$ $(a^2 < 1)$.

$\left[= \frac{2\pi}{a^2(a^2-1)} \text{ if } a^2 > 1\right]$.

Sol. Putting $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$; $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $\cos 2\theta = \frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right) = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$, we get

$$I = \int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \int_{|z|=1} \frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}{1 - a\left(z + \frac{1}{z}\right) + a^2} \cdot \frac{dz}{iz}.$$

$$= \frac{1}{2i} \int_{|z|=1} \frac{(z^4+1)\,dz}{z^2(z - az^2 - a + a^2 z)} = \frac{1}{2i} \int_{|z|=1} \frac{(z^4+1)}{z^2(z-a)(1-az)}\,dz$$

$$= \frac{1}{2i} \int_{|z|=1} f(z)\,dz, \text{ say.}$$

Now f has simple poles at $z = a$, $z = \frac{1}{a}$ and a second order pole at $z = 0$, of which the poles at $z = a$ and $z = 0$ lie within the unit circle $(\because a^2 < 1)$.

So $\mathrm{Res}(f; a) = \lim_{z \to a} (z-a) \frac{z^4+1}{z^2(z-a)(1-az)} = \frac{a^4+1}{a^2(1-a^2)}$

$\mathrm{Res}(f; 0) = \lim_{z \to 0} \frac{d}{dz}\left[z^2 f(z)\right]$.

$= \lim_{z \to 0} \frac{d}{dz} \frac{z^4+1}{(z-a)(1-az)}$