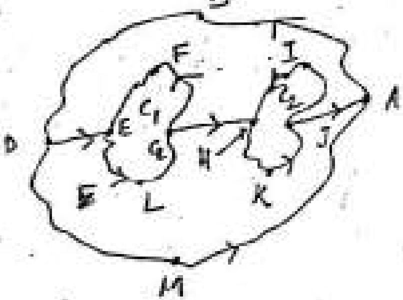
V. If $C_1, C_2, \ldots, C_n$ are simple closed curves no two of which have common point and if C is any simple closed curve which contains $C_1, C_2, \ldots, C_n$ in its interior, then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz,$$ provided f is analytic in the closed region bounded by these curves.



Proof We introduce three cuts DE, GH and JA joining C, C_1 and C_2. Since f is analytic in the closed region bounded by C, C_1, C_2, by Cauchy's Fundamental Theorem we get

$$\int_{ABDEFGHIJA} f(z)\,dz = 0 \quad \text{and} \quad \int_{DMAJKHGLED} f(z)\,dz = 0$$

i.e., $$\int_{ABD} f(z)\,dz + \int_{DE} f(z)\,dz + \int_{EFG} f(z)\,dz + \int_{GH} f(z)\,dz + \int_{HIJ} f(z)\,dz + \int_{JA} f(z)\,dz = 0 \quad \text{------} \; (1)$$

and $$\int_{DMA} f(z)\,dz + \int_{AJ} f(z)\,dz + \int_{JKH} f(z)\,dz + \int_{HG} f(z)\,d \ldots + \int_{GLE} f(z)\,dz + \int_{ED} f(z)\,dz = 0.$$

i.e., $$\int_{DMA} f(z)\,dz - \int_{JA} f(z)\,dz + \int_{JKH} f(z)\,dz - \int_{GH} f(z) \ldots + \int_{GLE} f(z)\,dz - \int_{DE} f(z)\,dz = 0 \quad \text{------} \; (2).$$

Adding (1) and (2) we get,

$$\left\{ \int_{ABD} f(z)\,dz + \int_{DMA} f(z)\,dz \right\} + \left\{ \int_{EFG} f(z)\,dz + \int_{GLE} f \ldots \right. + \left\{ \int_{HIJ} f(z)\,dz + \int_{JKH} f(z)\,dz \right\} = 0.$$

i.e., $$\int_{ABDMA} f(z)\,dz + \int_{EFGLE} f(z)\,dz + \int_{HIJKH} f(z)\,dz = 0$$

i.e., $$\oint_C f(z)\,dz + \int_{-C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

i.e., $$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz$$

Example Evaluate $\int_{|z|=1} \frac{dz}{z+2}$ and deduce that

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0.$$

Sol^n It is clear that the fn $f(z) = \frac{1}{z+2}$ is analytic within and on $|z|=1$. In fact $\frac{1}{z+2}$ has only one singularity at $z=-2$. So by Cauchy's Fundamental theorem we get

$$\int_{|z|=1} \frac{dz}{z+2} = 0.$$

Parametric eqn of the unit circle is $x = \cos\theta$, $y = \sin\theta$. So,

$$0 = \int_{|z|=1} \frac{dz}{z+2} = \int_0^{2\pi} \frac{1}{\cos\theta + i\sin\theta + 2}(-\sin\theta + i\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} \frac{(-\sin\theta + i\cos\theta)(\cos\theta + 2 - i\sin\theta)}{(\cos\theta+2)^2 + \sin^2\theta}\, d\theta$$

$$\int_0^{2\pi} \frac{-\sin\theta\cos\theta - 2\sin\theta + \sin\theta\cos\theta}{5+4\cos\theta}\, d\theta + i\int_0^{2\pi} \frac{\cos^2\theta + 2\cos\theta + \sin^2\theta}{5+4\cos\theta}\, d\theta.$$

Equating the imaginary parts we get

$$\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0.$$

i.e, $2\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0$ $\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x) \text{ if } f(2a-x) = f(x)\right]$

i.e, $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0.$

Ex Evaluate $\int_C \frac{dz}{z-\alpha}$, where C denotes any simple closed curve and α is an interior point. What is the value of the integral when α lies outside C.

Sol^n Let Γ be a circle lying within C with α as centre and radius r. Since $\frac{1}{z-\alpha}$ is analytic in the region bounded by C and Γ we get.

$$\int_C \frac{dz}{z-\alpha} = \int_\Gamma \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{ire^{i\theta}\, d\theta}{re^{i\theta}}, \text{ putting } z-\alpha = re^{i\theta},\ 0 \le \theta \le 2\pi$$

$= 2\pi i.$

If α is outside C, $\frac{1}{z-\alpha}$ is analytic within and on C so that by Cauchy's F. theorem

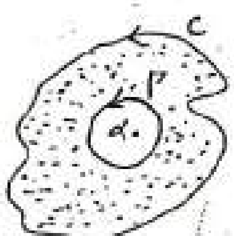
we get $\int_C \frac{dz}{z-\alpha} = 0$.

Cauchy's Integral Formula

Theorem Let f be analytic within and on a simple closed contour C and let α be any point inside C.

Then $f(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}\,dz$, ----- (1)

[where the integral is taken in the positive sense along C].



Proof Let Γ denote the circle $|z-\alpha| = r$, the radius of the circle is taken so small that Γ lies entirely within C. The function $\frac{f(z)}{z-\alpha}$ is clearly analytic in the closed annulus bounded by C and Γ

Hence $\oint_C \frac{f(z)}{(z-\alpha)}\,dz = \oint_\Gamma \frac{f(z)}{z-\alpha}\,dz$ ---- (2)

Now, $\oint_\Gamma \frac{f(z)}{z-\alpha}\,dz = \oint_\Gamma \frac{f(z)-f(\alpha)+f(\alpha)}{z-\alpha}\,dz$.

$= \oint_\Gamma \frac{f(z)-f(\alpha)}{z-\alpha}\,dz + f(\alpha)\oint_\Gamma \frac{dz}{z-\alpha}$

$= \int_\Gamma \frac{f(z)-f(\alpha)}{z-\alpha}\,dz + 2\pi i f(\alpha)\left[\because \oint \frac{dz}{z-\alpha} = 2\pi i\right]$

Hence from (2) we get

$$\left|\oint_C \frac{f(z)}{z-\alpha}\,dz - 2\pi i f(\alpha)\right| = \left|\oint_\Gamma \frac{f(z)-f(\alpha)}{z-\alpha}\,dz\right| \quad ---- (3).$$

Since f is continuous at $z=\alpha$; for given ε exists a $\delta(>0)$ such that $|f(z)-f(\alpha)| < \varepsilon$

$|z-\alpha| < \delta$.

Now we choose $r < \delta$ so that $|f(z)-f(\alpha)| < \varepsilon$ for all z on Γ. Also on Γ, $|z-\alpha| = r$. Hence on Γ we get $\left|\frac{f(z)-f(\alpha)}{z-\alpha}\right| < \frac{\varepsilon}{r}$.

Therefore we obtain from (3) using ML- that $\left|\oint_C \frac{f(z)}{z-\alpha}\,dz - 2\pi i f(\alpha)\right| \le \frac{\varepsilon}{r}\cdot 2\pi r =$

Since the l.h.s. is a constant term and arbitrary, it follows that

$\oint_C \frac{f(z)}{z-\alpha}\,dz - 2\pi i f(\alpha) = 0$ i.e., $f(\alpha) = \frac{1}{2\pi i}$

Note. The formula (1) is remarkable in the sen it expresses the value of an analytic func a point within a closed contour in term values on the contour.

Ex Evaluate $\oint_C \frac{z\,dz}{(9-z^2)(z+i)}$ where C is the circle $|z|=2$

Soln Let $f(z) = \frac{z}{9-z^2}$ which is analytic within and on C. So applying Cauchy's integral formula we get

$$\frac{1}{2\pi i}\int \frac{z\,dz}{(9-z^2)(z+i)}\,dz = f(-i)$$

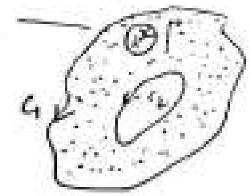
i.e, $$\int_C \frac{z\cdot dz}{(9-z^2)(z+i)} = 2\pi i \cdot \frac{-i}{9-(-i)^2} = 2\pi i\,\frac{-i}{9-(-i)^2} = \frac{\pi}{5}.$$

Cauchy's integral formula can be extended to multiply connected domains as shown in the following theorem.

Theorem If f is analytic in the closed annulus bounded by two closed contours C_1 and C_2 (C_2 lying wholly within C_1) and if α is any point in this annular region, then

$$f(\alpha) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-\alpha}\,dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-\alpha}\,dz$$

Proof Let $\Gamma : |z-\alpha| = r$, where r is so chosen that Γ lies entirely within the annulus determined by C_1 and C_2. The function $\frac{f(z)}{z-\alpha}$ is clearly analytic in the multiply connected region bounded by C_1, C_2 and Γ. Therefore,



$$\oint_{C_1} \frac{f(z)}{z-\alpha}\,dz = \oint_{C_2} \frac{f(z)}{z-\alpha}\,dz + \oint_{\Gamma} \frac{f(z)}{z-\alpha}\,dz.$$

i.e, $$\oint_{C_1} \frac{f(z)}{z-\alpha}\,dz - \oint_{C_2} \frac{f(z)}{z-\alpha}\,dz = \oint_{\Gamma} \frac{f(z)}{z-\alpha}\,dz.$$

From this step proceed as the preceding theorem.

Cauchy's Integral Formula For derivatives

Theorem Let f be analytic within and on a simple closed contour C. If α is any point interior to C then

$$f'(\alpha) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-\alpha)^2}\,dz \quad \text{---} (1)$$

Proof Let d be the lower bound of the distances of the point α from the contour C. If h denotes a complex number such that $|h|<d$, then the point $\alpha+h$ also lies within C. Therefore, by Cauchy's integral formula

$$f(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}\,dz \text{ and } f(\alpha+h) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha-h}\,dz.$$

Therefore,
$$\frac{f(\alpha+h)-f(\alpha)}{h} = \frac{1}{2\pi i}\oint_C f(z)\left\{\frac{1}{z-\alpha-h} - \frac{1}{z-\alpha}\right\}\frac{dz}{h}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)(z-\alpha-h)}\,dz$$

Hence
$$\left|\frac{f(\alpha+h)-f(\alpha)}{h} - \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^2}\,dz\right|$$

$$= \left|\frac{1}{2\pi i}\oint_C f(z)\left\{\frac{1}{(z-\alpha)(z-\alpha-h)} - \frac{1}{(z-\alpha)^2}\right\}dz\right|$$

$$= \frac{|h|}{2\pi}\left|\oint_C \frac{f(z)}{(z-\alpha)^2(z-\alpha-h)}\,dz\right| \quad \text{---- (2)}$$

Since f is continuous on C, it is bounded on C, so there exists a positive number M such that $|f(z)| \le M$ for all $z \in C$. Also by the definition of d, $|z-\alpha| \geq d$ $\forall z \in C$, and so $\frac{1}{|z-\alpha|} \le \frac{1}{d}$ for all $z \in C$. Again for all $z \in C$, $|z-\alpha-h| \geq |z-\alpha| - |h| \geq d - |h|$. Therefore on C,

$$\therefore \left|\frac{f(z)}{(z-\alpha)^2(z-\alpha-h)}\right| \le \frac{M}{d^2(d-|h|)}$$

Hence by ML-formula we get from (2)

$$\left|\frac{f(\alpha+h)-f(\alpha)}{h} - \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^2}\,dz\right| \le \frac{|h|}{2\pi}\cdot\frac{Ml}{d^2(d-|h|)}$$

$\rightarrow 0$ as $h \rightarrow 0$, where l is the length of

Consequently $\lim_{h\to 0} \frac{f(\alpha+h)-f(\alpha)}{h} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^2}$

i.e., $f'(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^2}\,dz$.

This proves the theorem.

Theorem Let f be analytic within and on a closed contour C. Then for any point α interior to C

$$f^{(n)}(\alpha) = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}\,dz \quad \text{---- (1)}$$

for $n = 0, 1, 2, \ldots\ldots$

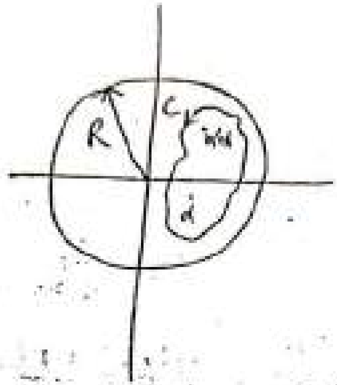
Proof We prove the theorem by mathematical induction. First we note that (1) is true for $n = 0, 1$. Now suppose that (1) is true for $n = m$ and prove it is true for $n = m+1$.

Let d be the lower bound of the distance of the point α from the contour C. If h is a complex number such that $|h| < d$ then

point $\alpha+h$ also lies within C.
Therefore we get from (1) for $n=m$

$$f^{(m)}(\alpha) = \frac{\lfloor m}{2\pi i}\oint \frac{f(z)}{(z-\alpha)^{m+1}}dz \text{ and}$$

$$f^{(m)}(\alpha+h) = \frac{\lfloor m}{2\pi i}\oint \frac{f(z)}{(z-\alpha-h)^{m+1}}dz$$



Then $$f^{(m)}(\alpha+h) - f^{(m)}(\alpha) = \frac{\lfloor m}{2\pi i}\oint f(z)\,\frac{(z-\alpha)^{m+1}-(z-\alpha-h)^{m+1}}{(z-\alpha)^{m+1}(z-\alpha-h)^{m+1}}dz \quad \text{--------- (2)}$$

Let $t = z-\alpha$. Then we get

$$(z-\alpha)^{m+1} - (z-\alpha-h)^{m+1} = t^{m+1} - (t-h)^{m+1}$$
$$= h\left[t^m + t^{m-1}(t-h) + t^{m-2}(t-h)^2 + \cdots + (t-h)^m\right]$$

So from (2) we get

$$\frac{f^{(m)}(\alpha+h) - f^{(m)}(\alpha)}{h} = \frac{\lfloor m}{2\pi i}\oint f(z)\,\frac{t^m + t^{m-1}(t-h)+\cdots+(t-h)^m}{t^{m+1}(t-h)^{m+1}}dz.$$

Now, $$\frac{f^{(m)}(\alpha+h) - f^{(m)}(\alpha)}{h} - \frac{\lfloor m+1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{m+2}}dz.$$

$$= \frac{\lfloor m}{2\pi i}\oint f(z)\left\{\frac{t^m + t^{m-1}(t-h)+\cdots+(t-h)^m}{t^{m+1}(t-h)^{m+1}} - \frac{m+1}{t^{m+2}}\right\}dz$$

$$= \frac{\lfloor m}{2\pi i}\oint_C f(z)\,\frac{t^{m+1} + t^m(t-h)+\cdots+t(t-h)^m - (m+1)(t-h)^{m+1}}{t^{m+2}(t-h)^{m+1}} \quad \text{------- (3)}$$

Also,
$$t^{m+1} + t^m(t-h) + \cdots + t(t-h)^m - (m+1)(t-h)^{m+1}$$
$$= \{t^{m+1} - (t-h)^{m+1}\} + (t-h)\{t^m - (t-h)^m\} + (t-h)^2\{t^{m-1} - (t-h)^{m-1}\} + \cdots + (t-h)^m\{t-(t-h)\}$$
$$= \{t-(t-h)\}\{t^m + t^{m-1}(t-h) + \cdots + (t-h)^m\}$$
$$+ (t-h)\{t-(t-h)\}\{t^{m-1} + t^{m-2}(t-h) + \cdots + (t-h)^{m-1}\}$$
$$+ \cdots + \{t-(t-h)\}(t-h)^m$$
$$= h\left[\{t^m + t^{m-1}(t-h) + \cdots + (t-h)^m\} + \{t^{m-1}(t-h) + t^{m-2}(t-h) \cdots + (t-h)^m\} + \cdots + (t-h)^m\right]$$

Now when $z\in C$ it follows that
$|t| = |z-\alpha| \geq d$, $|t-h| = |z-\alpha-h| \geq |z-\alpha| - |h| \geq d - |h|$
and $|t| = |z-\alpha| \leq 2R$, $|t-h| = |z-(\alpha+h)| \leq 2R$.
where R is the upperbound of the distances of the pt α from all pts of C.
Again since f is continuous on C, it is bounded on C and so there exists a positive number M such that $|f(z)| \leq M \;\forall\, z\in C$.
Therefore for $z \in C$ we get.

$$\left|\frac{t^{m+1} + t^m(t-h) + \cdots + t(t-h)^m - (m+1)(t-h)^{m+1}}{t^{m+2}(t-h)^{m+1}}\right|$$

$$\le |h| \frac{\{|t|^m + |t|^{m-1}|t-h| + \cdots + |t-h|^m\} + \{|t|^{m-1}|t-h| + \cdots + |t-h|^m\} + \cdots + |t-h|^m}{|t|^{m+2}|t-h|^{m+1}}$$

$$\le |h| \frac{(2R)^m N}{d^{m+2}(d-|h|)^{m+1}}$$, where N is the number of terms in the numerator.

Hence we get from (3) by ML-formula

$$\left| \frac{f^{(m)}(\alpha+h) - f^{(m)}(\alpha)}{h} - \frac{\lfloor m+1}{2\pi i} \oint \frac{f(z)}{(z-\alpha)^{m+2}} dz \right|$$

$$\le \frac{\lfloor m}{2\pi} |h| M (2R)^m N \frac{1}{d^{m+2}(d-|h|)^{m+1}}$$, where

L is the length of C.

Taking limit as $h \to 0$ we get

$$\lim_{h\to 0} \frac{f^{(m)}(\alpha+h) - f^{(m)}(\alpha)}{h} = \frac{\lfloor m+1}{2\pi i} \oint_C \frac{f(z)}{(z-\alpha)^{m+2}} dz.$$

i.e., $$f^{(m+1)}(\alpha) = \frac{\lfloor m+1}{2\pi i} \oint \frac{f(z)}{(z-\alpha)^{m+2}} dz.$$

Therefore by mathematical induction the theorem is true. This proves the theorem.

Note $f(x) = \frac{1}{2}x^2$ if $x \ge 0$

$= -\frac{1}{2}x^2$ if $x < 0$

Then $f'(x) = |x|$. So $f''(0)$ does not exists.

Theorem Let f be analytic in a domain D, all the derivatives of f exist and are analytic in D i.e., $f^{(1)}(z)$ $f^{(2)}(z), \ldots, f^{(n)}(z), \ldots$ all are analytic in D.

Proof Let $z_0 \in D$ and let C be a circle with at z_0 contained in D. Then for $n = 0, 1, 2, \ldots$

$$f^{(n)}(\alpha) = \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{n+1}} dz$$ $\forall\ \alpha$ interior to

Thus f has derivatives of all orders of z_0. Since z_0 is any point in D, the proved.

Note The situation is completely different in real function. There exist functions which a first derivative but no second.

Ex Show that $$\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i} \oint_C \frac{a^n e^{az}}{n!\, z^{n+1}} dz,$$

where C is any closed contour round the

Soln By Cauchy's integral formula for the derivative we get $$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-\alpha)^{n+1}} dz$$

Putting $\alpha = 0$ and $f(z) = e^{az}$ we obtain

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint_C \frac{e^{az}}{z^{n+1}} dz, \quad \ldots \ldots (1)$$

because $f(z) = e^{az}$ is analytic everywhere.

Also $f^{(n)}(z) = a^n e^{az}$ so that $f^{(n)}(0) = a^n$ .... (2)

From (1) and (2) we get

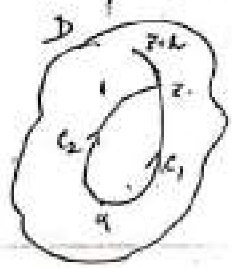
$$a^n = \frac{n!}{2\pi i} \oint_C \frac{e^{az}}{z^{n+1}} dz.$$

i.e., $$\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i} \oint_C \frac{a^n e^{az}}{n!\, z^{n+1}} dz = \frac{1}{2\pi i} \oint_C \frac{a^n e^{az}}{n!\, z^{n+1}} dz.$$

We shall now establish a theorem due to Morera, which is a sort of converse of Cauchy's Fundamental theorem.

Morera's Theorem

Statement: If f is continuous in a simply connected domain D and if $\int_\Gamma f(z)dz = 0$ for every closed rectifiable curve Γ in D then f is analytic in D.



Proof: Let α be any fixed and z a variable point in D. Let C_1, C_2 be any two rectifiable curves in D joining α and z. Then the curve consisting of C_1 and C_2 is a closed rectifiable curve in D so that by the given condition

$$\int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0 \quad \text{i.e.,} \int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

This shows that the integral of f is independent of the path so long as the path lies in D. We now define a function ϕ in D as

$$\phi(z) = \int_\alpha^z f(w)\,dw \text{ ..... (1)}$$

Definition of ϕ is justified because the integral (1) depends only on the upper limit z and not on the path joining α and z when α is fixed and $\phi(z)$ has a definite value for every $z \in D$. Then

$$\phi(z+h) = \int_\alpha^{z+h} f(w)\,dw.$$

Therefore $$\phi(z+h) - \phi(z) = \int_\alpha^{z+h} f(w)\,dw - \int_\alpha^z f(w)\,dw$$

$$= \int_\alpha^z f(w)\,dw + \int_z^{z+h} f(w)\,dw - \int_\alpha^z f(w)\,dw = \int_z^{z+h} f(w)\,dw \text{ ... (2)}$$

The integral (2) being independent of the path of integration we may take the same along the straight line joining z and $z+h$.

Now, $$\frac{\phi(z+h) - \phi(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} f(w)\,dw - f(z).$$

$$= \frac{1}{h}\int_z^{z+h} \{f(w) - f(z)\}\,dw \text{ ..... (3)}$$

Since by hypotheses f is continuous at z, for every $\varepsilon(>0)$ there exists a $\delta(>0)$ such that

$$|f(\omega)-f(z)| < \varepsilon \text{ whenever } |\omega - z| < \delta.$$

We choose $z+h$ in D such that $|h| < \delta$. Then for every point ω on the straight line joining z to $z+h$, we have $|f(\omega)-f(z)| < \varepsilon$.

Hence from (3) we get by ML-formula

$$\left|\frac{\varphi(z+h)-\varphi(z)}{h} - f(z)\right| = \frac{1}{|h|}\left|\int_z^{z+h}\{f(\omega)-f(z)\}d\omega\right|$$

$$\leq \frac{1}{|h|}\cdot\varepsilon|h| = \varepsilon \text{ for } 0<|h|<\delta.$$

This gives $\lim_{h\to 0}\frac{\varphi(z+h)-\varphi(z)}{h} = f(z)$.

Thus $\varphi'(z)$ exists at each point $z\in D$ and $\varphi'(z) = f(z)$. Therefore φ is analytic in D. Since f is the derivative of an analytic function, it follows that f itself is analytic in D. This proves the theorem.

Cauchy's Inequality

Statement If f is analytic within and on a circle C with centre α and radius r and if $|f(z)| \leq M$ $\forall z\in C$, M being a positive constant, then $|f^{(n)}(\alpha)| \leq \frac{M\,\underline{|n}}{r^n}$, $n = 0, 1, 2, \ldots$

Proof Since f is analytic within and on C and α is an interior point of C, we get

$$f^{(n)}(\alpha) = \frac{\underline{|n}}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-\alpha)^{n+1}} \quad \cdots\cdots (1)$$

Also on C, $|f(z)| \leq M$ and $|z-\alpha| = r$. Thus we get from (1)

$$|f^{(n)}(\alpha)| = \left|\frac{\underline{|n}}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}dz\right|$$

$$= \frac{\underline{|n}}{2\pi}\left|\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}dz\right|$$

$$\leq \frac{\underline{|n}}{2\pi}\,\frac{M}{r^{n+1}}\cdot 2\pi r \quad [\text{By ML-formula}]$$

i.e., $|f^{(n)}(\alpha)| \leq \frac{M\,\underline{|n}}{r^n}$.

This proves the theorem.

Definition A function of a complex variable that is analytic throughout the complex plane is called an integral or entire function.

Statement If f is an integral function and if $|f(z)| \le M$ for all z, M being a positive constant then f is a constant, i.e., a bounded integral function is constant.

Proof Let α be any point in the open complex plane and let $\Gamma : |z-\alpha| = r$. Then $|f(z)| \le M$ for all z on Γ no matter how large the radius r is. The function f is analytic within and on Γ and α is a point within Γ, so by Cauchy's integral formula for derivatives

$$f'(\alpha) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z-\alpha)^2}\,dz = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z-\alpha)^2}\,dz$$

Therefore,
$$|f'(\alpha)| = \left| \frac{1}{2\pi i} \oint_{C} \frac{f(z)}{(z-\alpha)^2}\,dz \right|$$
$$\le \frac{1}{2\pi} \frac{M}{r^2} \cdot 2\pi r = \frac{M}{r},$$ because on Γ, $|z-\alpha| = r$ and $|f(z)| \le M$ and by ML-formula.

Letting $r \to \infty$, we see that $f'(\alpha) = 0$. Since α is any point, it follows that $f'(z) = 0$ for all z.

Let $f(z) = u(x,y) + i\,v(x,y)$. Since $f(z)$ is an integral function, it follows that $f'(z) = u_x + i v_x$ and by Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$. Hence $f'(z) = 0 \ \forall z$ implies that $u_x = u_y = v_x = v_y = 0$ for all x, y. So $du = u_x dx + u_y dy = 0$ and $dv = v_x dx + v_y dy = 0$. This gives $u = \text{const.}$ and $v = \text{const.}$ Therefore f is a constant. This proves the theorem.

As an application of Liouville's theorem we prove the following:

Fundamental Theorem of Classical Algebra.

Statement If f is a polynomial of degree n with real or complex coefficients then the equation $f(z) = 0$ has at least one root.

Proof Let $f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ $(a_n \ne 0)$ be a polynomial of degree n. If possible, suppose that no value of z exists for which $f(z) = 0$. We shall show that this leads to a contradiction.

Since f is polynomial, it is an integral function. Also since $f(z) \ne 0$ for any z, it follows that $g(z) = \frac{1}{f(z)}$ is an integral function.

Since for $z \ne 0$, $f(z) = z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z} + a_n \right)$

it follows that $|f(z)| \ge |z|^n \left\{ |a_n| - \frac{|a_{n-1}|}{|z|} - \frac{|a_{n-2}|}{|z|^2} - \dots - \frac{|a_0|}{|z|^n} \right\}$

and so we get $|f(z)| \to \infty$ as $|z| \to \infty$. Therefore

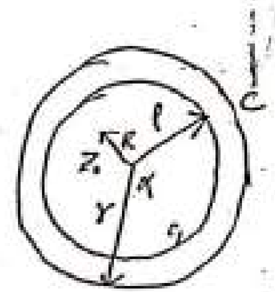
$|\varphi(z)| \to 0$ as $|z| \to \infty$. Hence φ is bounded. Thus φ, being a bounded integral function, is constant by Liouville's theorem. So f is also constant, which implies a contradiction because f is nonconstant when $a_n \neq 0$ and $n = 1, 2, 3, \ldots$. Therefore $f(z) = 0$ has at least one root. This proves the theorem.

Taylor's Theorem

Statement Let f be analytic in the interior of a circle C with centre α and radius r. Then at each point z interior to C,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n \quad \text{where } a_n = \frac{f^{(n)}(\alpha)}{n!}.$$

Proof Let z_0 be an arbitrary but fixed point within C and let $|z_0 - \alpha| = R$. We now choose a positive number ρ such that $R < \rho < r$.



Let C_1 denote the circle $|z-\alpha| = \rho$. Then C_1 lies entirely within C and z_0 is an interior point of C_1. Clearly f is analytic within and on C_1. Hence by Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z - z_0}\, dz,$$ the integral being taken in

Also, $|z_0-\alpha| = R$ and $|z-\alpha| = \rho$ for all $z \in C_1$. Further
$|z-z_0| = |(z-\alpha)-(z_0-\alpha)| \geq |z-\alpha| - |z_0-\alpha| = \rho - R$ for all $z \in C_1$.
i.e. $\frac{1}{|z-z_0|} \leq \frac{1}{\rho - R}$ for all $z \in C_1$.

Therefore,

$$|R_n| = \frac{|z_0-\alpha|^n}{2\pi}\left|\int_{C_1}\frac{f(z)}{(z-\alpha)^n(z-z_0)}dz\right|$$

$$\leq \frac{R^n}{2\pi}\cdot\frac{M}{\rho^n(\rho-R)}\cdot 2\pi\rho, \text{ by ML-formula}$$

$$= \frac{M\rho}{\rho-R}\left(\frac{R}{\rho}\right)^n \to 0 \text{ as } n\to\infty \left[\because \frac{R}{\rho}<1\right]$$

i.e., $\lim_{n\to\infty} R_n = 0$.

Hence from (1) it follows that

$$f(z_0) = \sum_{n=0}^{\infty}\frac{f^{(n)}(\alpha)}{n!}(z_0-\alpha)^n.$$

Since z_0 is an arbitrary point within C; for every z within C we get

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$$

$$= \sum_{n=0}^{\infty} a_n(z-\alpha)^n, \text{ where } a_n = \frac{f^{(n)}(\alpha)}{n!}$$

This proves the theorem.

<u>Note 1</u>. The power series representing f is called the Taylor series of f about the point $z=\alpha$. The Taylor series for f shows that if f is analytic in a nbd. of α then f can be represented in that nbd by a power series in $z-\alpha$ with a positive radius of convergence.

<u>Note 2</u>. Let f be analytic at α. Then there exists a circle $C: |z-\alpha| = r$ such that f is analytic within C. Then for each point z within C we have $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n$. The radius of the greatest circle within which the power series $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ ~~convergence~~ converges to $f(z)$ is the distance of the point α from the singular point of $f(z)$ which is nearest to α.

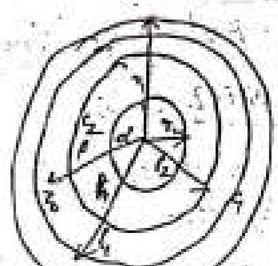
<u>Note 3</u> If f is an integral function then it has a Taylor expansion about the origin of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, valid for all z. This series is called <u>entire series</u>.

<u>Laurent's Theorem</u>

The following theorem which gives a generalisation of the Taylor's theorem is due to Laurent.

Statement Let f be a single valued analytic function on an annulus $D: r_2 < |z-\alpha| < r_1$. Then at each point $z \in D$, f can be represented by a series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n} \text{ ------ } (1)$$



Proof Let z_0 be an arbitrary point of D and let $|\alpha - z_0| = \rho$. We choose two positive numbers ρ_1, ρ_2 such that $r_2 < \rho_2 < \rho < \rho_1 < r_1$. Let $C_1 : |z-\alpha| = \rho_1$ and $C_2 : |z-\alpha| = \rho_2$. Then f is analytic in the closed annulus $\rho_2 \le |z-\alpha| \le \rho_1$, and the point z_0 lies inside the closed annulus.

Hence by Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}\,dz, \text{ --- } (2)$$

the integrals along C_1, C_2 being taken in the positive sense.

Now $\dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-\alpha)-(z_0-\alpha)}\,dz.$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-\alpha)\left(1-\frac{z_0-\alpha}{z-\alpha}\right)}\,dz.$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-\alpha}\cdot\frac{1-t^n+t^n}{1-t}\,dz, \quad t = \frac{z_0-\alpha}{z-\alpha}$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-\alpha}\left(1+t+t^2+\cdots+t^{n-1}+\frac{t^n}{1-t}\right)dz$$

$$= \sum_{m=0}^{n-1} \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-\alpha}\,t^m\,dz + \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-\alpha}\cdot\frac{t^n}{1-t}\,dz$$

$$= \sum_{m=0}^{n-1} (z_0-\alpha)^m \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-\alpha)^{m+1}}\,dz + \frac{(z_0-\alpha)^n}{2\pi i}\int_{C_1} \frac{f(z)}{(z-z_0)(z-\alpha)^n}\,dz$$

$$= \sum_{m=0}^{n-1} a_m (z_0-\alpha)^m + R_n, \text{ where } a_m = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-\alpha)^{m+1}}\,dz$$

and $R_n = \dfrac{(z_0-\alpha)^n}{2\pi i}\displaystyle\int_{C_1} \frac{f(z)}{(z-z_0)(z-\alpha)^n}\,dz.$

Since f is analytic on C_1, there exists a positive constant M such that $|f(z)| \le M$. Also $|z-\alpha| = \rho_1$ on C_1. Then $|z-z_0| = |z-\alpha+\alpha-z_0| \ge |z-\alpha| - |\alpha - z_0|$ $= \rho_1 - \rho$. So. $|R_n| = \left|\dfrac{(z_0-\alpha)^n}{2\pi i}\displaystyle\int_{C_1} \frac{f(z)}{(z-z_0)(z-\alpha)^n}\,dz\right|$

$$\le \frac{\rho^n}{2\pi}\cdot\frac{M}{(\rho_1-\rho)\rho_1^n}\cdot 2\pi\rho_1 = \frac{M\rho_1}{\rho_1-\rho}\left(\frac{\rho}{\rho_1}\right)^n \to 0$$

$n \to \infty$ $\left[\because \frac{\rho}{\rho_1} < 1\right]$

i.e., $R_n \to 0$ as $n \to \infty$.

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}\,dz = \sum_{n=0}^{\infty} a_n (z_0-\alpha)^n \quad \ldots\ldots (3),$$

where $a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-\alpha)^{n+1}}\,dz$.

Next $-\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z_0-z}\,dz$

$$= \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z_0-\alpha)-(z-\alpha)}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z_0-\alpha)\left(1-\frac{z-\alpha}{z_0-\alpha}\right)}\,dz$$

$$= \frac{1}{2\pi i}\cdot\frac{1}{z_0-\alpha}\int_{C_2} f(z)\cdot\frac{1-t^n+t^n}{1-t}\,dz, \text{ where}$$

$t = \frac{z-\alpha}{z_0-\alpha}$.

$$= \frac{1}{2\pi i}\cdot\frac{1}{z_0-\alpha}\int_{C_2} f(z)\left(1+t+t^2+\cdots+t^{n-1}+\frac{t^n}{1-t}\right)dz$$

$$= \frac{1}{2\pi i}\cdot\frac{1}{z_0-\alpha}\sum_{m=0}^{n-1}\int_{C_2} f(z)\,t^m\,dz + \frac{1}{2\pi i}\cdot\frac{1}{z_0-\alpha}\int_{C_2} f(z)\,\frac{t^n}{1-t}\,dz$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{n-1}(z_0-\alpha)^{-m-1}\int_{C_2} f(z)(z-\alpha)^m\,dz + \frac{1}{2\pi i}\,\frac{1}{(z_0-\alpha)^n}\int_{C_2}\frac{f(z)\,(z-\alpha)^n}{z_0-z}\,dz$$

put $m+1 = n$

$$= \sum_{n=1}^{n} b_n (z_0-\alpha)^{-n} + R_n', \text{ where } b_n = \frac{1}{2\pi i}\int_{C_2} f(z)(z-\alpha)^{n-1}\,dz,$$

$$R_n' = \frac{1}{2\pi i}\,\frac{1}{(z_0-\alpha)^n}\int_{C_2} f(z)\,\frac{(z-\alpha)^n}{z_0-z}\,dz.$$

Now on C_2 $|z-\alpha| = \rho_2$, $|z_0-z| = |z_0-\alpha+\alpha-z| \geq |z_0-\alpha|-|z-\alpha|$ $= \rho-\rho_2$. Also f is analytic on C_2, so there exists a positive constant N such that $|f(z)| \leq N$ for all $z \in C_2$. Then

$$|R_n'| = \left|\frac{1}{2\pi i}\,\frac{1}{(z_0-\alpha)^n}\int_{C_2} f(z)\cdot\frac{(z-\alpha)^n}{z_0-z}\,dz\right|$$

$$\leq \frac{1}{2\pi}\cdot\frac{1}{\rho^n}\cdot\frac{N\rho_2^n}{\rho-\rho_2}\cdot 2\pi\rho_2$$

$$= \frac{N\rho_2}{\rho-\rho_2}\left(\frac{\rho_2}{\rho}\right)^n \to 0 \text{ as } n\to\infty \quad \left[\because \frac{\rho_2}{\rho}<1\right]$$

i.e., $R_n' \to 0$ as $n\to\infty$.

Hence we obtain $-\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-z_0}\,dz = \sum_{n=1}^{\infty} b_n (z_0-\alpha)^{-n}$ $\ldots\ldots(4)$, where $b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-\alpha)^{-n+1}}\,dz$.

Combining (3) and (4) we get from (2)

$$f(z_0) = \sum_{n=0}^{\infty} a_n (z_0-\alpha)^n + \sum_{n=1}^{\infty} b_n (z_0-\alpha)^{-n}$$

Since z_0 is an arbitrary point of D, it follows that for all $z \in D$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n}, \text{ where}$$

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz, \quad b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-\alpha)^{-n+1}}\, dz$$

and C_1, C_2 denote respectively the circles $|z-\alpha| = \rho_1$, $|z-\alpha| = \rho_2$, $r_2 < \rho_2 < \rho_1 < r_1$ and $\rho_2 < |z-\alpha| < \rho_1$.

The series given in (2) representing f is called the Laurent's series for f in the annular region D. We note that the functions $\frac{f(z)}{(z-\alpha)^{n+1}}$ and $\frac{f(z)}{(z-\alpha)^{-n+1}}$ are both analytic in the annular region D. Hence it follows that the coefficients a_n, b_n are also given by $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{n+1}}\, dz$,

$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{-n+1}}\, dz$, where C is any circle with centre α and lying in D. Now changing n to $-n$ in a_n we find

$$a_{-n} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{-n+1}}\, dz = b_n$$

Hence the Laurent's series expansion of f can be written as $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n$, $r_2 < |z-\alpha| < r_1$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{n+1}}\, dz$, $n = 0, \pm 1, \pm 2, \ldots$

and C is any circle with centre α and lying in D and the integral along C is taken in the positive sense.

If f is analytic within and on C_1, since $-n+1 \le 0$ for $n = 1, 2, \ldots$, we see that functions $\frac{f(z)}{(z-\alpha)^{-n+1}}$ are analytic within and on C. Hence $b_n = 0$ for $n = 1, 2, \ldots$ and the Laurent's series reduces to Taylor's series.

Sometimes the theorem does not provide a method for calculating the coefficients, but since the Laurent's and Taylor's expansion of a function about a point are unique in a region around the point, we can compute the coefficients by an easier method as shown below.

<u>Example</u> Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent's series valid for (i) $|z| < 1$, (ii) $1 < |z| < 3$, (iii) $|z| > 3$, (iv) $0 < |z+1| < 2$.

<u>Sol:</u> (i) When $|z| < 1$

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

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$$= \frac{1}{2}\left[(1+z)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[(1 - z + z^2 - z^3 + \cdots) - \frac{1}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)\right]$$

$$= \frac{1}{2}\left[\frac{2}{3} - \left(1-\frac{1}{9}\right)z + \left(1-\frac{1}{27}\right)z^2 - \left(1-\frac{1}{81}\right)z^3 + \cdots\right]$$

$$= \frac{1}{2}\left[\frac{2}{3} - \frac{8}{9}z + \frac{26}{27}z^2 - \frac{80}{81}z^3 + \cdots\right]$$

$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$, which is the Laurent's expansion for f in $|z|<1$ and this coincides with Taylor's expansion.

(ii) When $1<|z|<3$, $f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$.

For $|z|>1$ we get

$$\frac{1}{2(z+1)} = \frac{1}{2z\left(1+\frac{1}{z}\right)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

For $|z|<3$ we get

$$\frac{1}{2(z+3)} = \frac{1}{6\left(1+\frac{z}{3}\right)} = \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Hence the Laurent's series of f valid in $1<|z|<3$ is $f(z) = \cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6}$

$$+ \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

(iii) When $|z|>3$, $f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$

$$= \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right)$$

$$= \frac{1}{2z}\left[\frac{2}{z} - \frac{8}{z^2} + \frac{26}{z^3} - \cdots\right]$$

$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \cdots$, which is the Laurent's series for $f(z)$ in $|z|>3$.

(iv) When $0<|z+1|<2$, $f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$.

f is analytic except at $z=-1$, $z=-3$. Hence f can be expressed in a Laurent's series in the annulus $0<|z+1|<2$ in positive and negative powers of $z+1$. Since we want to expand the function f in powers of $z+1$, we need not care for the term $\frac{1}{2(z+1)}$. Now

$$\frac{1}{z(z+3)} = \frac{1}{z(z+1+2)} = \frac{1}{4\left(1+\frac{z+1}{2}\right)}$$

$$= \frac{1}{4}\left(1+\frac{z+1}{2}\right)^{-1}$$

$$= \frac{1}{4}\left[1-\frac{1}{2}(z+1)+\frac{1}{4}(z+1)^2-\frac{1}{8}(z+1)^3+\cdots\right]$$

Therefore the Laurent's expansion of f is

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \frac{1}{32}(z+1)^3 + \cdots$$

in $0 < |z+1| < 2$

Example Show that $\cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n+\frac{1}{z^n}\right)$,

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cosh(2\cos\theta)\, d\theta$.

Sol: The function $\cosh\left(z+\frac{1}{z}\right)$ is analytic for all finite z except at $z=0$. Hence it is analytic in the annular region $r \le |z| \le R$, no matter how small the positive number r may be or how large the no. R may be. We can therefore expand f in a Laurent's series in the annulus $0 < |z| < \infty$ in the form $\cosh\left(z+\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$, where

$$a_n = \frac{1}{2\pi i}\int_C \frac{\cosh\left(z+\frac{1}{z}\right)}{z^{n+1}}\, dz, \quad b_n = \frac{1}{2\pi i}\int_C \frac{\cosh\left(z+\frac{1}{z}\right)}{z^{-n+1}}\, dz,$$

where C is any circle with origin as centre. We take C to be the circle $|z|=1$. $z = e^{i\theta}$ on C where θ varies from 0 to 2π.

Hence $a_n = \frac{1}{2\pi i}\int_0^{2\pi} \cosh\left(e^{i\theta}+e^{-i\theta}\right) \frac{ie^{i\theta}}{e^{i(n+1)\theta}}\, d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\, (\cos n\theta - i\sin n\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta - \frac{i}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\sin n\theta\, d\theta$$

If we put $\theta = 2\pi - \phi$, we find that the second integral vanishes and so

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta$$

and $b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos(-n\theta)\, d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta$$

$$= a_n, \quad n = 1, 2, \ldots$$

Therefore, $\cosh\left(z+\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$

$= a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$, in $0 < |z| < \infty$.

Zeros and singularities of an analytic function

defn A point α is called a regular point or an ordinary point of a function f if f is analytic at α, otherwise α is called a singular point or a singularity of f.

defn Let f be an analytic fn with a singular point at $z = \alpha$. If there exists a nbd. of α which contains no other singular points of f except α, then α is called an isolated singular point or an isolated singularity of the fn f.

Suppose that α is an isolated singularity of the function f. In this case f can be expanded in a Laurent's series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n}$$

in a domain $0 < |z-\alpha| < r$, where r is the distance of the point α from the nearest singularity of f other than α itself. The portion of the series involving negative powers of $z-\alpha$ i.e, $\sum_{n=1}^{\infty} b_n (z-\alpha)^{-n}$ is called the principal part of f at α, while the series of nonnegative powers of $z-\alpha$ i.e, $\sum_{n=0}^{\infty} a_n (z-\alpha)^n$ is called the regular part of f at α.

There are three cases to be considered.

Case I Suppose that all the coffs. b_n are zero. We then call $z = \alpha$, a removeable singularity of f because we can make f regular when $|z-\alpha| < r$ by suitably defining its value at α. Singularity of this type is of little importance.

Case II The principal part of f at α contains a finite number of terms only. Then f is said to have a pole at $z = \alpha$. If $b_m (m \geq 1)$ is the last nonvanishing coffs. in the principal part then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \frac{b_1}{z-\alpha} + \frac{b_2}{(z-\alpha)^2} + \cdots + \frac{b_m}{(z-\alpha)^m}$$

$(0 < |z-\alpha| < r)$ and the pole is said to be of order m (a simple, a double in the case $m = 1, 2$).

Case III The principal part of f at $z=\alpha$ contains infinitely many non zero terms. The point $z = \alpha$

is then called an essential singularity of f.
In this case

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n}$$

where the last series does not terminate but is convergent for all z in $0 < |z-\alpha| < r$.

Ex Let $f(z) = \frac{\sin z}{z}$, $z \neq 0$

$= 0$, $z = 0$.

This function is analytic everywhere except at $z=0$. The Laurent's expansion about $z=0$ has the form

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

Since no negative power of z appears the pt. $z=0$ is a removable singularity. If we define $f(z) = 1$ at $z=0$, the modified function becomes analytic at $z=0$.

Ex The function $f(z) = \frac{z^2 - 2z + 3}{z-2} = 2 + (z-2) + \frac{3}{z-2}$ $(z \neq 2)$ has a simple pole at $z=2$.

Ex The function $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$ has an essential singularity at $z=0$.

Note By def. a pole is an isolated singularity. If a singularity is nonisolated then also it is an essential singularity.

Theorem The function f has a pole of order m at α iff f can be expressed in the form in some nbd. of α, where ψ is analytic at α and $\psi(\alpha) \neq 0$.

Proof Let α be a pole of f of order m. In some nbd. of α, f has Laurent expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{j=1}^{m} b_j (z-\alpha)^{-j}, \text{ where } b_m \neq 0$$

$$= \phi(z) + \sum_{j=1}^{m} b_j (z-\alpha)^{-j}, \text{ say}$$

$$= \frac{(z-\alpha)^m \phi(z) + b_m + \sum_{j=1}^{m-1} b_j (z-\alpha)^{m-j}}{(z-\alpha)^m}$$

$$= \frac{\psi(z)}{(z-\alpha)^m}, \text{ where } \psi(z) = (z-\alpha)^m \phi(z) + b_m + \cdots$$

Clearly ψ is analytic at $z=\alpha$ and $\psi(\alpha) = b_m \neq 0$.
Next we suppose that in some nbd. of $z=\alpha$ we can write $f(z) = \frac{\psi(z)}{(z-\alpha)^m}$, where ψ is analytic

$z = \alpha$ and $\psi(\alpha) \neq 0$.

Now we expand ψ in the Taylor's series around α to get $\psi(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$ where $a_0 \neq 0$ because $\psi(\alpha) \neq 0$.

So,

$$f(z) = \frac{\sum_{n=0}^{\infty} a_n (z-\alpha)^n}{(z-\alpha)^m}$$

$$= \frac{a_0}{(z-\alpha)^m} + \frac{a_1}{(z-\alpha)^{m-1}} + \cdots + \frac{a_{m-1}}{z-\alpha} + \sum_{n=m}^{\infty} a_n (z-\alpha)^{n-m},$$

which is the Laurent's expansion of f around α. Since in the principal part the coeff. of $(z-\alpha)^{-m}$ is nonvanishing, it follows that α is a pole of f of order m. This proves the theorem.

<u>Definition</u> Let α be a regular point of an analytic fn f and if $f(\alpha) = 0$ then α is called a zero of f.

<u>Defn</u> The point $z = \alpha$ is called a zero of f of order or multiplicity m if in some nbd. of α f can be expanded in a Taylor series of the form $f(z) = \sum_{n=m}^{\infty} a_n (z-\alpha)^n$ where $a_m \neq 0$.

<u>Theorem</u> The pt. α is a zero of f of order m iff in some nbd. of α f can be expressed in the form $f(z) = (z-\alpha)^m \varphi(z)$, where φ is analytic at α, and $\varphi(\alpha) \neq 0$.

<u>Proof</u> Let α be a zero of f of order m. Then in a nbd. of α we can expand f as

$$f(z) = \sum_{n=m}^{\infty} a_n (z-\alpha)^n \quad \text{where } a_m \neq 0.$$

Then $f(z) = (z-\alpha)^m \sum_{n=m}^{\infty} a_n (z-\alpha)^{n-m} = (z-\alpha)^m \varphi(z)$,

where $\varphi(z) = \sum_{n=m}^{\infty} a_n (z-\alpha)^{n-m}$. Clearly φ is analytic at $z = \alpha$ and $\varphi(\alpha) = a_m \neq 0$.

Next we suppose that in some nbd. of α, $f(z) = (z-\alpha)^m \varphi(z)$ where φ is analytic at α and $\varphi(\alpha) \neq 0$.

Now we expand φ in a Taylor series around α to get $\varphi(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$. Since $\varphi(\alpha) \neq 0$, it follows that $a_0 \neq 0$. So in some nbd. of $z = \alpha$ we obtain

$$f(z) = (z-\alpha)^m \sum_{n=0}^{\infty} a_n (z-\alpha)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-\alpha)^{n+m}$$

$$= \sum_{p=m}^{\infty} A_p (z-\alpha)^p, \quad \text{where } A_p = a_{p-m} \text{ and } A_m = a_0 \neq 0.$$

So $z = \alpha$ is a zero of f of order m. This proves the theorem.

<u>Theorem.</u> If α is a zero of an analytic function f then there exists a nbd. of α which contains no other zero of f, unless the fn is identically zero. In otherwords ...

[redacted]

are isolated points unless the f^n is identically zero.

<u>Proof</u> Let α be a zero of order m of an analytic function f. We can, therefore, write $f(z) = (z-\alpha)^m \varphi(z)$, where φ is analytic at α and $\varphi(\alpha) \neq 0$.

Let $\varepsilon = \frac{1}{2}|\varphi(\alpha)| > 0$. Since φ is cont. at α, there exists a positive number δ such that

$$|\varphi(z) - \varphi(\alpha)| < \varepsilon = \tfrac{1}{2}|\varphi(\alpha)| \text{ for } |z-\alpha| < \delta. \text{ Therefore}$$

for $|z-\alpha| < \delta$

$$|\varphi(z)| = |\varphi(z) - \varphi(\alpha) + \varphi(\alpha)|$$
$$\geq |\varphi(\alpha)| - |\varphi(z) - \varphi(\alpha)|$$
$$> |\varphi(\alpha)| - \tfrac{1}{2}|\varphi(\alpha)| = \tfrac{1}{2}|\varphi(\alpha)| > 0$$

i.e. $\varphi(z) \neq 0$ in $|z-\alpha| < \delta$.

Since $f(z) = (z-\alpha)^m \varphi(z)$, it follows that f can not vanish in $0 < |z-\alpha| < \delta$. Hence α is an isolated zero of f. Thus the zeros of an analytic f^n are isolated points.

<u>Note</u> Incidentally we prove that if φ is analytic at α and $\varphi(\alpha) \neq 0$, then there exists a nbd. of α in which φ does not vanish.

<u>Relation between zeros of and poles.</u>

<u>Theorem</u> The point α is a pole of order m of a f^n f iff. it is a zero of order m of the f^n $\frac{1}{f}$.

<u>Proof</u> Suppose first that f has a pole at $z=\alpha$ of order m. Then in some nbd. of $z=\alpha$ we can write $f(z) = \varphi(z)\{z-\alpha\}^m = \varphi(z)(z-\alpha)^{-m}$, where φ is analytic at $z=\alpha$ and $\varphi(\alpha) \neq 0$. Therefore $\frac{1}{f(z)} = \dots$ where $\psi(z) = \frac{1}{\varphi(z)}$ is analytic at α and $\psi(\alpha) = \frac{1}{\varphi(\alpha)}$ ...

So α is a zero of $\frac{1}{f(z)}$ of order m.

Conversely let $z=\alpha$ be a zero of $\frac{1}{f}$ of order m, we can write $\frac{1}{f(z)} = (z-\alpha)^m g(z)$ in some nbd. where g is analytic at α and $g(\alpha) \neq 0$. Then

$$f(z) = \frac{1}{(z-\alpha)^m g(z)} = \frac{h(z)}{(z-\alpha)^m}, \quad \text{where } h(z) = 1/g(z) \text{ is}$$

analytic at α and $h(\alpha) = \frac{1}{g(\alpha)} \neq 0$. Hence α is a pole of f of order m. This proves the theorem.

<u>Corollary</u> If a function f has essential singularity at α then $\frac{1}{f}$ has also an essential singularity at α.

<u>Proof</u> If possible let α be a regular pt. of $\frac{1}{f}$ & α be not a zero of $1/f$. Then α is a regular pt. of f, which contradicts the nature of α.

If possible let α be a regular pt. of

α be a zero of order m of $1/f$. Then α is a pole of order m of f, which contradicts the nature of α.

If possible let α be a pole of order m of $1/f$. Then α is a zero of order m of f which also contradicts the nature of α.

Hence the only possibility that remains for α is to be an essential singularity of $1/f$.

Behaviour of a function near a pole

<u>Theorem</u> If α is a pole of the f^n^ f then $|f(z)| \to \infty$ as $z \to \alpha$.

<u>Proof</u>. Let α be a pole of f of order m. Then in some nbd. of α we can write $f(z) = \frac{\varphi(z)}{(z-\alpha)^m}$, where φ is analytic at α and $\varphi(\alpha) \neq 0$. Therefore $|f(z)| = \frac{|\varphi(z)|}{|z-\alpha|^m}$. Since φ is continuous at α (φ being analytic at α), for the positive number $\varepsilon = \frac{1}{2}|\varphi(\alpha)|$ we can find a positive number δ such that $|\varphi(z) - \varphi(\alpha)| < \frac{1}{2}|\varphi(\alpha)|$ for $|z-\alpha| < \delta$.

Therefore $|\varphi(z)| = |\varphi(z) - \varphi(\alpha) + \varphi(\alpha)|$

$$\geq |\varphi(\alpha)| - |\varphi(z) - \varphi(\alpha)|$$

$$> |\varphi(\alpha)| - \frac{1}{2}|\varphi(\alpha)| = \frac{1}{2}|\varphi(\alpha)| \quad \text{for } |z-\alpha| < \delta.$$

Hence $|f(z)| > \frac{\frac{1}{2}|\varphi(\alpha)|}{|z-\alpha|^m}$ for $0 < |z-\alpha| < \delta$.

Now let G be any +ve ~~integer~~ number, however large, then $|f(z)| > G$ whenever $\frac{\frac{1}{2}|\varphi(\alpha)|}{|z-\alpha|^m} > G$

i.e., whenever $0 < |z-\alpha|^m < \min\left\{\delta^m, \frac{\frac{1}{2}|\varphi(\alpha)|}{G}\right\}$

i.e., whenever $0 < |z-\alpha| < \min\left\{\delta, \left(\frac{\frac{1}{2}|\varphi(\alpha)|}{G}\right)^{1/m}\right\} = \delta_1$

This shows that $|f(z)| \to \infty$ as $z \to \alpha$ and the theorem is proved.

<u>Limit points of zeros and poles.</u>

<u>Theorem</u> The limit pt. of the zeros of an analytic f^n^ is an essential singularity of the f^n^, unless the function is identically zero.

<u>Proof</u> Let α be a limit pt. of the zeros of a f^n^ f. Then an infinity of zeros of f lies in every deleted nbd. of α.

If possible, let α be a regular pt. of f. Then f is cont. at α. So for given $\varepsilon(>0)$ there exists a $\delta(>0)$ such that $|f(z) - f(\alpha)| < \varepsilon$ for $|z-\alpha| < \delta$

Since there is an infinity of zeros of f in $0<|z-\alpha|<\delta$, for all these zeros we must have $|f(\alpha)|<\varepsilon$. Since $\varepsilon(>0)$ is arbitrary, we get $f(\alpha)=0$. Hence α is a zero of f which is impossible unless $f\equiv 0$, because the zeros are isolated points. Hence α is not a regular pt. of f and so α must be a singularity of f.

If possible, let α be a pole of f. Then $|f(z)|\to\infty$ as $z\to\alpha$. i.e., given any positive number G we can find a number $\eta(>0)$ such that

$$|f(z)|>G \quad \text{- - - - - - -} (1)$$

for all z in $0<|z-\alpha|<\eta$.

Since the deleted nbd. $0<|z-\alpha|<\eta$ contains an infinity of zeros of f, the inequality (1) can not hold for all z in $0<|z-\alpha|<\eta$. Hence α can not be a pole of f.

Thus α is an essential singularity of f unless $f\equiv 0$. This proves the theorem.

<u>Theorem</u> The limit pt. of poles of an analytic function f is a non-isolated essential singularity of f.

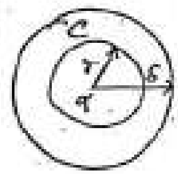
<u>Proof</u> Let α be a limit pt. of the poles of f.

is evidently the pt. $z=1$. So $z=1$ is an essential singularity of $\sin\frac{1}{z-1}$. Hence $z=-1, 1$ are two essential singularities of f.

Riemann's Theorem

Statement: If a fn. f is bounded and analytic throughout a domain $0<|z-\alpha|<\delta$, then either f is analytic at α or else α is a removable singularity of f.

Proof Under the given hypothesis f can be represented in the Laurent's series in the given domain about α in the form



$$f(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n+\sum_{n=1}^{\infty} b_n(z-\alpha)^{-n}.$$

If C denote the circle $|z-\alpha|=r$, $r<\delta$, the coefficients of $\frac{1}{(z-\alpha)^n}$ are given by

$$b_n=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\alpha)^{-n+1}}\,dz=\frac{r^n}{2\pi}\int_0^{2\pi} e^{in\theta} f(\alpha+re^{i\theta})\,d\theta,$$

$n=1,2,3,\ldots$, putting $z-\alpha=re^{i\theta}$.

Since f is bounded, there is a positive number M such that $|f(z)|<M$. Hence $|b_n|<Mr^n$, $n=1,2,3,\ldots$. Since the coefficients b_n are independent of r and r can be chosen arbitrary small, $b_n=0$ for $n=1,2,3,\ldots$. Thus Laurent's series for f reduces to

$$f(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n \text{ in } 0<|z-\alpha|<\delta.$$

This shows that either f is analytic at α or else α is a removable singularity of f. This proves the theorem.

Theorem of Weierstrass and Casorati

Statement If α is an isolated essential singularity of the fn f, then given any positive numbers r and ε and any finite complex number c, there is a point z in $0<|z-\alpha|<r$ at which $|f(z)-c|<\varepsilon$. If $c=\infty$ then $\lim_{n\to\infty} f(z_n)=\infty$ for a sequence $\{z_n\}$ tending to α.

Proof (i) Let $c=\infty$. We note that there is not a single nbd. of the pt. α in which f is bounded. For, otherwise the pt. α, by Riemann's Theorem, would be a removable singularity. This means that for each positive integer n there is a point z_n in the annulus K_n: $0<|z-\alpha|<\frac{1}{n}$ such that $|f(z_n)|>n$ i.e., $z_n\to\alpha$, $f(z_n)\to\infty$ as $n\to\infty$.

(ii) Let $c\neq\infty$. If possible, let us suppose that the theorem is not true. Then there is a +ve number ε_0 and a +ve number r_0 such that for all $z : 0<|z-\alpha|<r_0$ we have $|f(z)-c| \geq \varepsilon_0$ ....(1)

Let

$$g(z) = \frac{1}{f(z)-c} \quad \cdots\cdots (2)$$

From (1) and (2) we get $|g(z)| \leq \frac{1}{\varepsilon_0}$ in $0<|z-\alpha|<r_0$. Since α is an isolated singular point of f, it is at worst an isolated singular point for g too. Also g is bounded in $0<|z-\alpha|<r_0$ (because $|g(z)| \leq \frac{1}{\varepsilon_0}$) and so by Riemann's theorem $z=\alpha$ is a removable singularity of g. Therefore $\lim_{z\to\alpha} g(z) = \beta$ (say), exists. From (2) we get

$f(z) = c + \frac{1}{g(z)}$ in $0<|z-\alpha|<r_0$ and so $\lim_{z\to\alpha} f(z)$ exists finitely or infinitely according as $\beta \neq 0$ or $\beta = 0$. i.e., the point α is either a removable singularity or a pole of f which contradicts the hypothesis of the theorem. This proves the theorem.

The point at infinity (∞).

The behaviour of a f? f at ∞ is considered by making the substitution $z = \frac{1}{\zeta}$ and exami[ning] $f(1/\zeta)$ at $\zeta = 0$.

We say that f is regular or has a pol[e or] an essential singularity at $z=\infty$ if $f(1/\zeta)$ [has] same property at $\zeta = 0$.

For example the function $f(z) = z^2$ has a d[ouble pole] at $z=\infty$.

Example Show that infinity is a simple ze[ro of]

$$f(z) = \frac{az^2+bz+c}{lz^3+mz^2+nz+p}.$$

Sol^n We put $z = \frac{1}{\zeta}$ to get

$$f(1/\zeta) = \zeta \cdot \frac{a+b\zeta+c\zeta^2}{l+m\zeta+n\zeta^2+p\zeta^3}.$$

Since $\zeta = 0$ is a simple zero for the f? f[(1/ζ)], that $z=\infty$ is a simple zero for the functi[on].

Theorem A f? which is analytic everyw[here incl.] the pt. at infinity is a constant.

Proof Since f is analytic for all finite [z, by] Taylor's theorem we get $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and the series converges for all finite z. $f(1/\zeta) = \sum_{n=0}^{\infty} a_n \frac{1}{\zeta^n}$. Since $f(1/\zeta)$ is regular