Introduction

There is no real number x which satisfies the polynomial equation $x^2+1=0$. To permit solutions of this and similar equations, the set of complex numbers is introduced.

From a strictly logical point of view it is desirable to define a complex number as an ordered pair (a,b) of real numbers a and b subject to certain operational definitions. These definitions are as follows, where all letters represent real

A. Equality $(a,b)=(c,d)$ iff $a=c, b=d$

B. Sum $(a,b)+(c,d)=(a+c,\ b+d)$.

C. Product $(a,b)\cdot(c,d)=(ac-bd,\ ad+bc)$

$m(a,b)=(ma,\ mb)$.

Also we denote by i the ordered pair $(0,1)$ and we identify the real number a with the ordered pair $(a,0)$.

Now from the above definition we see that

$(a,b)=(a,0)+(0,b)=a(1,0)+b(0,1)=a\cdot 1+b\cdot i$
$=a+ib$

Further we see that $i^2=(0,1)\cdot(0,1)$

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Complex plane or Argand plane.

$= (0\cdot1,\ 0\cdot1 + 1\cdot0) = (-1, 0) = -1$.

The Complex Plane or Argand Plane

Let us consider two mutually perpendicular axes $X'OX$ and $Y'OY$, called x-axis and y-axis on a plane.

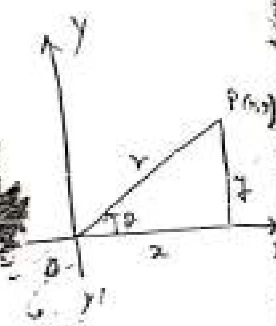
Since a complex number $x+iy$ can be considered as an ordered pair of real numbers, we can represent such numbers by points in the xy plane called the complex plane or Argand plane. To each complex number $z = x+iy$ there corresponds one and only one point (x,y) in the plane and conversely to each point (x,y) in the plane there corresponds one and only one complex number $z = x+iy$. Because of this, we often refer to the complex number z as the point z.

Some times we refer to x and y as the real and imaginary parts of $z = x+iy$ respectively and so the x-axis and y-axis are some times called the real and imaginary axes respectively. The complex plane is often called the z-plane.

The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the complex plane is given by

$$|z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

Polar form of complex numbers



If P is a point in the complex plane corresponding to the complex number (x,y) or $x+iy$ then we see from the figure that $x = r\cos\theta$, $y = r\sin\theta$ where $r = \sqrt{x^2+y^2} = |x+iy|$ is called the modulus or absolute value of $z = x+iy$ (denoted by $|z|$) and θ, called the amplitude, or argument of $z = (x+iy) = x+iy$ (denoted by $\arg z$), is the angle which line OP makes with the positive x-axis i.e., $\theta = \arg z = \tan^{-1}\frac{y}{x}$. If we restrict the angle θ in $0 \le \theta < 2\pi$, then it is called the principal argument of z.

Now it follows from the figure that $z = x+iy$ $= r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$, which is called the polar form of the complex number and r and θ are called the polar coordinates.

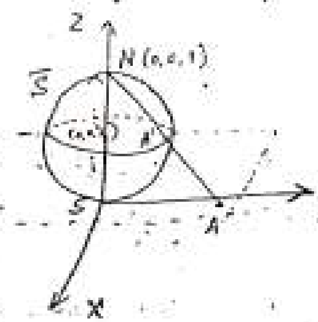
The point at infinity

In the Euclidean geometry of the plane "points at infinity" do not occur; two straight lines intersect in a point except in the case when the lines are parallel. It is, ~~however~~ however, usual to postulate, at a later stage, that there exists an infinite number of points at infinity, each being defined as the point of intersection of a pencil of parallel lines.

Now if we take a pencil of curves through the point $z = c$ in the Argand plane, the transformation $z' = \frac{1}{z}$ turns them into a pencil of curves through the point $z' = \frac{1}{c}$, provided that c is not zero. To avoid the difficulty of this exceptional case, we now postulate that there is a single point at infinity in the Argand plane; this point at infinity is defined to be the point corresponding to the origin in the transformation $z' = \frac{1}{z}$.

The nature of the Argand plane at the point at infinity is made much clearer by the use of Riemann's spherical representation of complex numbers, which depends on stereographic pro[jection].

Let C be the complex plane and consider a unit diameter sphere $\bar{S}$ tangent to C at $z=0$. The diameter NS is perpendicular to C and we call points N and S the north and south poles of $\bar{S}$. Corresponding to any point A on C we construct line NA intersecting $\bar{S}$ at point A'.



Thus to each point of the complex plane C there corresponds one and only one point of the sphere $\bar{S}$ and we can represent any complex number by a point on the sphere. For completeness we say that the point N itself corresponds to the "point at infinity" of the plane. The set of all points of the complex plane including the point at infinity is called the entire complex plane, the entire z-plane or the extended complex plane.

The above method for mapping the plane on to the sphere is called stereographic projection. The sphere is sometimes called the Riemann sphere.

Analytic Functions.

<u>Def:</u> A neighbourhood of a point $z_0 \in C$ is the set of all points z such that $|z-z_0| < r$ where r is some positive number i.e., the set of all points lying in the disc with centre z_0 and radius r.

A deleted nbd. of z_0 is a nbd. of z_0 in which the pt. z_0 is omitted i.e., $0 < |z-z_0| < r$.

<u>Def:</u> A pt. z_0 is called a <u>limit pt.</u> of a set S in the complex plane, if every deleted nbd. of z_0 contains at least a pt. of S.

A limit point may or may not belong to the set.

<u>Def:</u> A point $z_0 \in S$ is called an <u>interior pt.</u> of the set S if there exists a nbd. of z_0 contained entirely within S.

<u>Def:</u> A pt. $z_0 \in S$ is called an <u>isolated pt.</u> of S if there exists a nbd. of z_0 which does not contain any pt. of S other than z_0 itself.

<u>Def:</u> A set S in the complex plane is said to be <u>open</u> if it consists only of interior points.

<u>Def:</u> A set S is said to be <u>closed</u> if every limit point of S belongs to S or if S has no limit pt.

Alternatively, a set S is said to be <u>closed</u> if its complement is open.

<u>Example</u> The open disc $|z-a| < r$ is an open set and the closed disc $|z-a| \leq r$ is a closed set.

<u>Example</u> It should be observed that there exist sets which are neither open nor closed: the set consisting of the point $z=1$ and all pts for which $|z| < 1$

<u>Def:</u> The set of all limit points of a set S is called the <u>derived set</u> or the <u>first derived</u> of S and is denoted by S'.

<u>Def:</u> The union of a set S and its derived set S' is called the <u>closure</u> of S and is denoted by $\bar{S}$ or $Cl(S)$.

<u>Def:</u> A <u>boundary point</u> of a set S is a pt. nbd. of which contains at least one pt. of S and at least one pt. not of S.

<u>Def:</u> A set S is said to be <u>bounded</u> if there is a positive number M such that $|z| \leq M$ for all $z \in S$.

If there exists no such no. M then the set is said to be unbounded.

defn A set which is both bounded and closed is sometimes called compact.

definition A set is called connected if and only if any two of its points can be joined by polygon all of whose points belong to the set.

defn An open connected set is called a domain or an open region. The closure of an open region is called a closed-region.

Whenever we use the word region without qualifying it, we shall mean open region or a domain.

defn The equation $z = z(t) = x(t) + iy(t)$ where $x(t)$ and $y(t)$ are real continuous functions of the real variable t, defined in the interval $a \le t \le b$, determines a set of points in the complex plane which we call a continuous arc.

The equation $z = z(t) = x(t) + iy(t)$ determines a simple arc if $t_1 \neq t_2$ implies $z(t_1) \neq z(t_2)$ ...

$z = z(t)$ is a simple closed curve (S.C.C.) if $t_1 < t_2$ and $z(t_1) = z(t_2)$ implies $t_1 = a$, $t_2 = b$.

Simple arcs and simple closed curves are often called Jordan arcs and Jordan curves.

A simple example of a Jordan arc is the polygonal arc which consists of a finite number of line segment joined end to end.

The equation $z = \cos t + i \sin t$, $0 \le t \le 2\pi$ is a simple closed curve which is the unit circle $|z| = 1$.

defn A region or a domain R is called simply connected if any simple closed curve which lies in R can be contracted to a point without leaving R.

Alternatively, a region R is said to be simply connected if every simple closed curve lying within it encloses only points of the region.

A region which is not simply connected is called multiply connected.

The circular disc $|z| < R_1$ is simply connected but the circular ring $r_1 < |z| < r_2$ is multiply connected.

Assumed Results

1. The Jordan Curve Theorem

A simple closed J-curve divides the plane into two domains which have the curve as common boundary.

2 - Bolzano - Weierstrass Theorem

If a set is bounded and contains infinitely many points then it possesses at least one limit pt.

3. The neighbourhood of $z=\infty$.

A nbd. of $z=\infty$ consists of the set exterior of a circle in the finite complex plane with centre at the origin.

Function of a complex variable.

A fn f defined on a set $S \subset \mathbb{C}$ is a rule which assigns to each $z \in S$ a complex number w. The number w is called a value of f at z and is denoted by $f(z)$. i.e., $w=f(z)$.

The set, S is called the domain of definition of f.

A function is single valued on a set S if it has just one value corresponding to each $z \in S$. If more than one value correspond to each value of z we say that the function is multiple valued.

Whenever we speak of a function we mean single valued, unless otherwise stated.

Let D be a domain in the z-plane and let $w=f(z)$ be defined in D. To each $z=x+iy \in D$ is then assigned a definite value $w=u+iv$. Then the real and imaginary parts u and v of $f(z)$ are real functions of two real variables x and y : $u=u(x,y)$, $v=v(x,y)$. Conversely, any two such functions always define one complex function $f=u+iv$ of $z=x+iy$.

Limit and Continuity

defn. Let $w=f(z)$ be defined in a domain D except perhaps at the point z_0 of D. A complex number l is said to be the limit of f as z tends to z_0, symbolically $\lim_{z \to z_0} f(z) = l$, if for given $\varepsilon(>0)$ there exists a $\delta(>0)$ such that for all $z \in D$ for which $0<|z-z_0|<\delta$ we get $|f(z)-l|<\varepsilon$.

If no such number l exists we say that $\lim_{z \to z_0} f(z)$ does not exist.

Note that z is allowed to approach z_0 in an arbitrary manner, not just from some particular

The limit is clearly independent of the path by which z-approaches z_0.

Theorem. A necessary & suff condition that the function $f = u + iv$ may tend to $\alpha + i\beta$ as $z = x+iy$ tends to $z_0 = a+ib$ is that $u(x,y) \to \alpha$ and $v(x,y) \to \beta$ as $(x,y) \to (a,b)$.

Proof We first suppose that $\lim_{z \to z_0} f(z) = l$, where $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$, $l = \alpha + i\beta$. Then for given $\varepsilon (>0)$ there exists a $\delta (>0)$ such that

$$|f(z) - l| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta$$

i.e, $|(u-\alpha) + i(v-\beta)| < \varepsilon$ whenever

$$0 < |(x-a) + i(y-b)| < \delta.$$

i.e, whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon$.

Since for any complex number z $|Re(z)| \le |z|$ and $|Im(z)| \le |z|$ we get from above that

$$|u(x,y) - \alpha| < \varepsilon \text{ and } |v(x,y) - \beta| < \varepsilon$$

Whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. These imply that

$$\lim_{(x,y)\to(a,b)} u(x,y) = \alpha \text{ and } \lim_{(x,y)\to(a,b)} v(x,y) = \beta.$$

This proves the necessary part.

Next we suppose that $\lim_{(x,y)\to(a,b)} u(x,y) = \alpha$ and $\lim_{(x,y)\to(a,b)} v(x,y) = \beta$. Then for given $\varepsilon (>0)$ we can find a $\delta (>0)$ such that

$$|u(x,y) - \alpha| < \frac{\varepsilon}{2} \text{ and } |v(x,y) - \beta| < \frac{\varepsilon}{2}$$

whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. Now for $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ i.e., for $0 < |(x-a) + i(y-b)| < \delta$. i.e. for $0 < |z - z_0| < \delta$ we get

$$|f(z) - l| = |(u-\alpha) + i(v-\beta)|$$
$$\le |u-\alpha| + |v-\beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies that $\lim_{z \to z_0} f(z) = l$. This proves the sufficient part and the theorem.

Definition Let f be defined in a domain D except perhaps at the point $z_0 \in D$. The function f is said to tend to ∞ as z tends to z_0 if for any real number K, however large, there is a $\delta(>0)$ such that $|f(z)| > K$ whenever $0 < |z - z_0| < \delta$. In symbol we write $\lim_{z \to z_0} f(z) = \infty$.

Defn. If f is defined for $|z| > R$ for some $R > 0$ then $\lim_{z\to\infty} f(z) = c$ is defined as follows: if for every $\varepsilon (>0)$ there exists a number R_0 such that $|f(z) - c| < \varepsilon$

for $|z| > R_0$.

Defn We say that $\lim_{z\to\infty} f(z) = \infty$ if for each number $K(>0)$ there exists a number $R_0(>0)$ such that $|f(z)| > K$ whenever $|z| > R_0$.

Defn The function $w = f(z)$ is said to be continuous at z_0 if $f(z_0)$ is defined and $\lim_{z\to z_0} f(z) = f(z_0)$.

A fn f is said to be continuous in a region R if it is continuous at all points of the region.

Note To examine the continuity at $z = \infty$ we replace z by $\frac{1}{\zeta}$ and examine the continuity of $f(1/\zeta)$ at $\zeta = 0$.

Theorem If f is continuous in a bounded closed region R then f is bounded in R.

Proof If possible suppose that f is not bounded in R. Then for each positive integer n $\exists$ a pt. $z_n \in R$ such that $|f(z_n)| > n$. Since the region R is bounded, by Bolzano-Weierstrass theorem the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$. Since R is closed, if $\lim_{k\to\infty} z_{n_k} = z_0$ then $z_0 \in R$. Since f is continuous in R, it is cont. at z_0 and so $f(z_{n_k}) \to f(z_0)$ as $k \to \infty$. Since $|f(z_n)| > n$ for all values of n, it follows that for $n_k > N$

$$|f(z_0)| = |f(z_0) - f(z_{n_k}) + f(z_{n_k})|$$
$$\geq |f(z_{n_k})| - |f(z_0) - f(z_{n_k})|$$
$$> |f(z_{n_k})| - 1 \qquad [\because |f(z_0) - f(z_{n_k})| < 1 \text{ for } n_k > N]$$
$$> n_k - 1,$$

which implies that f is not defined finitely at z_0 — a contradiction to the fact that f is continuous in R. Therefore f is bounded in R.

This proves the theorem.

Differentiability (differentiability)

Defn Let f be a function defined in a domain D of the complex plane. If $z_0 \in D$ and if $\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ ... exists, we denote this limit by $f'(z_0)$ and call it the derivative of f at the point z_0.

If $f'(z_0)$ exists finitely then f is said to be differentiable at z_0. Equivalently

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$
$$= \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

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If f is differentiable at each point of D, we say that f is differentiable in D.

. We stress once again that the limit (1) exists means that the limit exists and is same along whatever path z approaches z_0.

Theorem If f is differentiable at z_0 then it is continuous at z_0.

Proof Since f is differentiable at z_0,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}.$$

Now $f(z_0+\Delta z) - f(z_0) = \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \cdot \Delta z$.

and so
$$\lim_{\Delta z \to 0} \{f(z_0+\Delta z) - f(z_0)\} = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \times \lim_{\Delta z \to 0} \Delta z.$$

$$= f'(z_0) \times 0 = 0.$$

Therefore $\lim\limits_{\Delta z \to 0} f(z_0+\Delta z) = f(z_0)$ and so f is continuous at z_0.

Example If $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$.

prove that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any straight line, but $f'(0)$ does not exist.

Solution Let $z \to 0$ along any straight line $y = mx$. Then

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{(x,y)\to(0,0)} \frac{xy^2(x+iy)}{(x^2+y^4)(x+iy)}$$

$$= \lim_{x \to 0} \frac{m^2x^3}{x^2+m^4x^4} = \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0.$$

Now let $z \to 0$ along $x = y^2$. Then

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{(x,y)\to(0,0)} \frac{xy^2(x+iy)}{(x^2+y^4)(x+iy)}$$

$$= \lim_{y \to 0} \frac{y^4}{y^4+y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}.$$

This shows that $f'(0)$ does not exist.

Cauchy – Riemann Equations or C-R Equations.

Theorem A necessary condition that the fn $w = f(z) = u(x,y) + iv(x,y)$ may be differentiable at the point $\zeta = \xi + i\eta$ is that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ at (ξ, η) i.e. $u_x(\xi,\eta) = v_y(\xi,\eta)$ and $u_y(\xi,\eta) = -v_x(\xi,\eta)$

Proof Suppose $f'(\zeta)$ exists. Then

$$f'(\zeta) = \lim_{z \to \zeta} \frac{f(z)-f(\zeta)}{z-\zeta} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta z}. \quad \ldots (1)$$

Where Δz is the increment of z from ζ and

Δw is the corresponding increment of $w = f(z)$.

Since $f'(\zeta)$ exists the above limit must exist for all modes of approach of Δz to zero. i.e, for all modes of approach of the point (x,y) to (ξ,η) and all the limiting values must be same.

Let $z \to \zeta$ along a line parallel to the real axis. Then ~~...~~ $y = \eta$ and $x \to \xi$ i.e, $\Delta y = 0$ and $\Delta z = \Delta x \to 0$. Therefore from (1) we get

$$f'(\zeta) = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right)$$

$$= \lim_{x \to \xi} \left[\frac{u(x,\eta) - u(\xi,\eta)}{x - \xi} + i \frac{v(x,\eta) - v(\xi,\eta)}{x - \xi} \right].$$

Since from the existence of limit of a complex function the existence of the limits of its real and imaginary parts follows, we see that at the point (ξ,η) there exist partial derivatives with respect to x of $u(x,y)$ and $v(x,y)$ and we have

$$f'(\zeta) = u_x(\xi,\eta) + i v_x(\xi,\eta) \quad \cdots\cdots (2)$$

Next let $z \to \zeta$ along a line parallel to the imaginary axis so that $x = \xi$ and $y \to \eta$ i.e, $\Delta x = 0$ and $\Delta z = i\Delta y \to 0$. Therefore from (1)

$$f'(\zeta) = \lim_{\Delta y \to 0} \left[\frac{1}{i} \frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta y} \right]$$

$$= \lim_{y \to \eta} \left[-i \frac{u(\xi,y) - u(\xi,\eta)}{y - \eta} + \frac{v(\xi,y) - v(\xi,\eta)}{y - \eta} \right]$$

$$= -i u_y(\xi,\eta) + v_y(\xi,\eta) \quad \cdots\cdots (3)$$

~~The~~ Therefore on comparing (2) and (3) and equating real and imaginary parts we obtain

$$u_x(\xi,\eta) = v_y(\xi,\eta), \quad u_y(\xi,\eta) = -v_x(\xi,\eta).$$

<u>Note</u> The differential equations $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$ are known as Cauchy-Riemann equations.

<u>Note</u> If $f'(z)$ exists, where $f(z) = u(x,y) + iv(x,y)$ then $f'(z) = u_x + iv_x = v_y - iu_y$.

<u>Example</u> Let $f(z) = |z|^2$. Find the nature of C-R equations.

Here $f(z) = |z|^2 = x^2 + y^2$ so that $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. Then $u_x = 2x$, $u_y = 2y$, $v_x = 0 = v_y$. Thus C-R equations are not satisfied unless $x = y = 0$ and $f'(z)$ can not therefore, exist if $z \neq 0$.

Following example shows that the valid-

Of C-R equations at a point is not suffient to ensure the existance of the derivative at that point.

Ex $f(z) = \frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}$, $z \neq 0$

$= 0$, $z = 0$

Show that though C-R equations are satisfied at $(0,0)$, $f'(0)$ does not exist.

Solution Here $u(x,y) = \frac{x^3-y^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$.

and $v(x,y) = \frac{x^3+y^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$.

Now $u_x(0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x^3/x^2}{x} = 1$.

$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{-y^3/y^2}{y} = -1$.

$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{x^3/x^2}{x} = 1$,

and $v_y(0,0) = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{y^3/y^2}{y} = 1$.

Therefore, $u_x(0,0) = v_y(0,0)$, $u_y(0,0) = -v_x(0,0)$.

Thus C-R equations are satisfied at the origin.

Now,

$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$

$= \lim_{(x,y)\to(0,0)} \left[\frac{\frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}}{x+iy} \right]$

$= \lim_{(x,y)\to(0,0)} \frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$

Letting $(x,y) \to (0,0)$ along the line $y = mx$ we get

$f'(0) = \lim_{x\to 0} \frac{x^3(1-m^3)+ix^3(1+m^3)}{x^2(1+m^2)\,x\,(1+im)}$

$= \frac{1-m^3+i(1+m^3)}{(1+m^2)(1+im)}$, which is different for different values of m. Thus $f'(0)$ does not exist.

Ex Let $f(z) = \sqrt{|xy|}$. Show that $f'(0)$ does not exist although the C-R equations are satisfied at the origin.

Theorem A single valued continuous function $w = f(z) = u(x,y) + iv(x,y)$ is differentiable in a region R, if the four partial derivatives u_x, u_y, v_x, v_y exist, are continuous and satisfy C-R equations at each point of R.

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Proof We are to show that $f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ exists at each point of R. Let $z = x+iy$ be any point of R. Since u_x, u_y, v_x, v_y exist and are continuous at (x,y), $u(x,y)$ and $v(x,y)$ are differentiable at (x,y). Therefore,

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$

$$= u_x \Delta x + u_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \text{ where}$$

$\epsilon_1 \to 0$, $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Similarly

$$\Delta v = v_x \Delta x + v_y \Delta y + \eta_1 \Delta x + \eta_2 \Delta y, \text{ where}$$

$\eta_1, \eta_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Now $\Delta w = \Delta u + i\Delta v$

$$= u_x \Delta x + u_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y + i[v_x \Delta x + v_y \Delta y + \eta_1 \Delta x + \eta_2 \Delta y] \quad \cdots (1)$$

Since u_x, u_y, v_x, v_y satisfy C-R equations $u_x = v_y$, $u_y = -v_x$, then (1) becomes

$$\Delta w = u_x(\Delta x + i\Delta y) + v_x(i\Delta x - \Delta y) + (\epsilon_1 + i\eta_1)\Delta x + (\epsilon_2 + i\eta_2)\Delta y$$

$$= u_x \Delta z + i v_x(\Delta x + i\Delta y) + (\epsilon_1 + i\eta_1)\Delta x + (\epsilon_2 + i\eta_2)\Delta y$$

$$= u_x \Delta z + i v_x \Delta z + (\epsilon_1 + i\eta_1)\Delta x + (\epsilon_2 + i\eta_2)\Delta y$$

So, $\frac{\Delta w}{\Delta z} = u_x + i v_x + (\epsilon_1 + i\eta_1)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\eta_2)\frac{\Delta y}{\Delta z}$

Now, $\left|(\epsilon_1 + i\eta_1)\frac{\Delta x}{\Delta z}\right| = |\epsilon_1 + i\eta_1| \left|\frac{\Delta x}{\Delta z}\right| \le |\epsilon_1| + |\eta_1| \to 0$ as $\Delta z \to 0$.

and $\left|(\epsilon_2 + i\eta_2)\frac{\Delta y}{\Delta z}\right| = |\epsilon_2 + i\eta_2| \left|\frac{\Delta y}{\Delta z}\right| \le |\epsilon_2| + |\eta_2| \to 0$ as $\Delta z \to 0$.

Therefore, proceeding to the limit as $\Delta z \to 0$ in (2) we get $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x + i v_x$ i.e, $f'(z)$ exists and is equal to $u_x + i v_x$. Since z is any point of R we thus conclude that f is differentiable in R.

Definition A function f defined in a domain D is said to be an analytic function in D or analytic in D if f has a finite derivative at each point of D.

The terms "regular" and "holomorphic" are also used in stead of analytic. This definition is due to Cauchy, - one of the founders of the theory of complex functions.

Definition A function f is said to be analy

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at a pt z_0 if it is analytic in an neighbourhood of z_0 i.e., if there exists a nbd. of z_0 at all points of which $f'(z)$ exists finitely.

If f is not analytic at a point z_0 then z_0 is called a singular point or singularity of f.

Definition If f is differentiable in a domain D except at a finite number of points in D, we say that f is analytic in D except at these points.

Defn If f is analytic for all finite z, it is said to be an integral or entire function.

The functions $f(z)=e^z$, $f(z)=a_0+a_1z+\cdots+a_n z^n$ are integral functions.

Theorem If $u+iv$ is analytic in D then the C-R equations hold in D. If u and v have continuous first order partial derivatives in D and the C-R equations hold in D, then f is analytic in D.

Ex Let $f=u+iv$ be analytic in a domain D. Show that f is constant in D if any one of the following conditions hold: (1) $f'(z)\equiv 0$ in D, (2) $Re\{f(z)\}$ = Constant in D, (3) $Im\{f(z)\}$ = Constant in D, (4) $|f(z)|$ = Const in D, (5) $\arg f(z)$ = Const in D.

Soln (1) Let $f'(z)\equiv 0$ in D. Then $f'=u_x+iv_x=0$ in D so that $u_x=v_x=0$. Since f is analytic in D, C-R equations hold and so $u_x=v_y=0$, $u_y=-v_x=0$ in D. Hence in D, $du=u_xdx+u_ydy=0$ and $dv=v_xdx+v_ydy=0$ and so u = constant and v = constant. Therefore f is a constant.

(2) Let $Re\{f(z)\}$ = const. in D. Then u = const in D so that $u_x=u_y=0$ in D. Since f is analytic in D, C-R equations hold and so $u_x=v_y=0$, $u_y=-v_x=0$ in D. Hence in D, $du=u_xdx+u_ydy=0$ and $dv=v_xdx+v_ydy=0$ and so u = constant, v = constant in D. Therefore f is a constant.

(3) Let $Im\{f(z)\}$ = Constant in D. Then v = constant in D so that $v_x=v_y=0$ in D. Since f is analytic in D, C-R equations hold in D and so $u_x=v_y=0$, $u_y=-v_x=0$ in D. Hence $du=u_xdx+u_ydy=0$ and $dv=v_xdx+v_ydy=0$ and so u = constant, v = constant in D.

(4) Let $|f(z)|$ = Constant. Then u^2+v^2 = Constant so that

$$uu_x+vv_x=0 \quad \cdots (i)$$

and

$$uu_y+vv_y=0 \quad \cdots (ii)$$

Since f is analytic in D, C-R equations hold in D i.e., $u_x = v_y$, $u_y = -v_x$ so from (ii) we get

$vu_x - uv_x = 0$ . . . (iii)

From (i) and (iii) $(u^2+v^2)u_x = 0$ i.e. $u_x = 0$ because otherwise $u = v = 0$ and f becomes constant.

Again from (i) and (iii) we get $(u^2+v^2)v_x = 0$ i.e. $v_x = 0$

Also by C-R equations we get $u_x = v_y = 0$, $u_y = -v_x = 0$

Hence $du = u_x dx + u_y dy = 0$, $dv = v_x dx + v_y dy = 0$ and so u = constant, v = constant in D. Therefore f is constant in D.

(5). Let $\arg(f(z))$ = constant in D. Then $\frac{v}{u} = c$, a constant. So $v = cu$ and hence $v_x = cu_x$, $v_y = cu_y$. Since f is analytic in D, $f' = u_x + iv_x$ $= v_y - iu_y$.

So $f' = u_x + iv_x = u_x + icu_x = u_x(1+ic)$ - - - (i)

and $f' = v_y - iu_y = v_y - \frac{i}{c}v_y = v_y\left(1-\frac{i}{c}\right)$

$= u_x\left(1-\frac{i}{c}\right)$, . . . (ii) $[\because u_x = v_y]$

From (i) and (ii) we get $1+ic = 1-\frac{i}{c}$

i.e. $c^2 = -1$, i.e. $c = \pm i$.

Let $c = -i$. Then $v_x = -iu_x$, $v_y = -iu_y$. So

$f' = u_x + iv_x = u_x + i(-iu_x) = 2u_x$ and

$f' = v_y - iu_y = -2iu_y$ which is a contradiction because a real quantity can not be equal to a purely imaginary quantity. So $c \neq -i$.

Let $c = i$ Then (i) we get $f' = u_x(1+i^2) = 0$ So by f is constant in D.

Complex Integration

Def: A curve Γ in the complex plane is a continuous complex valued function $z = z(t) = x(t) + iy(t)$ defined on a real interval $a \le t \le b$, and we write $\Gamma : z = z(t)$, $a \le t \le b$.

The points $z_0 = z(a)$ and $z_1 = z(b)$ are called the end points of Γ. $z(a)$ being called the initial point and $z(b)$ the terminal point.

If $z(a) = z(b)$ i.e. if the initial and the terminal points coincide, the curve Γ is called a closed curve.

The real variable t is called the parameter of the curve, and the equation $z = z(t)$ is called the parametric equation of the curve.

In calling $z_0 = z(a)$ the initial point and $z_1 = z(b)$ the terminal point of a curve Γ with the equation $z = z(t)$, $a \le t \le b$, we describe an orientation of Γ.

This means that a pt. $z' = z(t') \in \Gamma$ is regarded as preceding a point $z'' = z(t'') \in \Gamma$ if $z' \neq z''$ and $t' < t''$. From this it follows that a curve Γ may be thought of having two orientations according as t varies from a to b or b to a. The curve differing from Γ only by direction in which it is traversed will be denoted by $-\Gamma$.

Def: If for more than one value of t we get the same pt. z then z is called a multiple point of the curve.

Rectifiable arcs:

Let $\Gamma : z = z(t)$, $a \le t \le b$, be any Jordan arc. By a partition of $[a, b]$ we mean a set of points $P = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$ satisfying $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$. Then the interval $[t_{k-1}, t_k]$ is called the kth subinterval of P and we write $\Delta t_k = t_k - t_{k-1}$ so that $\sum_{k=1}^{n} \Delta t_k = b - a$. The collection of all possible partitions of $[a, b]$ will be denoted by $P[a, b]$.

Let $P = \{a = t_0, t_1, t_2, \ldots, t_{n-1}, t_n = b\}$ be any partition of the interval $[a, b]$. We write $z_r = z(t_r)$ so that corresponding to a partition P of $[a, b]$ we get the points $z_0, z_1, \ldots, z_{r-1}, z_r, \ldots, z_n$ on the curve.

Construct the sum $S_P = \sum^{n} |z_k - z_{k-1}|$ so that corresponding to every partition of the interval $[a,b]$ there exists a sum S_p. Clearly S_p denotes the length of the polygon inscribed which is obtained by drawing straight lines from z_0 to z_1, from z_1 to z_2 and so on. Taking into account all possible partitions of $[a,b]$ we get an aggregate $\{S_p\}$. The curve Γ is said to be <u>rectifiable</u> if the set $\{S_p\}$ is bounded for all partitions P of $[a,b]$. Further if the curve Γ is rectifiable, the least upper bound of the set $\{S_p\}$ is defined to be the length of the curve Γ. Thus if L denotes the length of Γ, then $L = \sup\{S_p : P \in \mathcal{P}[a,b]\}$

If the set $\{S_p\}$ is unbounded, the curve Γ is called nonrectifiable.

<u>Regular Curves (Arcs)</u> [or Smooth Curves]

A simple curve (Jordan arc) defined by $z = x(t) + i y(t)$, $a \le t \le b$, is called a <u>regular arc</u> or <u>regular curve</u> if the derivatives $\dot{x}(t)$ and $\dot{y}(t)$ exist and are continuous and they do not vanish simultaneously over the whole interval.

<u>Theorem</u> A regular arc is rectifiable and its length L is given by $L = \int_a^b |\dot{z}(t)|\,dt$.

<u>Theorem</u> A Jordan arc which consists of a finite number of regular arcs is rectifiable, its length being the sum of the lengths of the regular arcs forming it.

<u>definition</u> A simple curve is called a <u>contour</u> if it consists of a finite number of regular arcs.

<u>definition</u> A simple closed curve is called closed contour if it consists of a finite number of regular arcs.

Evidently a contour is rectifiable.

<u>Example</u> $z(t) = \cos t + i \sin t = e^{it}$, $0 \le t \le 2\pi$ is contour.

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Complex Integration

Let Γ be a rectifiable arc whose equation is $z = z(t) = x(t) + iy(t)$, $a \le t \le b$. Let f be any complex function of z defined on Γ. Let $P = \{a = t_0, t_1, t_2, \ldots, t_{k-1}, t_k, \ldots, t_n = b\}$ be a partition of $[a,b]$. Corresponding to the partition P, Γ is divided into n smaller arcs $\sigma_k = \widehat{z_{k-1} z_k}$ $(k = 1, 2, \ldots, n)$ where $z_k = z(t_k)$, $k = 0, 1, 2, \ldots, n$.

Also let $\zeta_k = \xi_k + i\eta_k = z(\tau_k)$, $t_{k-1} \le \tau_k \le t_k$, be an arbitrary point σ_k. We form the sum

$$S_P = \sum_{k=1}^{n} (z_k - z_{k-1}) f(\zeta_k).$$

Now if $\lim\limits_{\|P\| \to 0} S_P = \lim\limits_{\|P\| \to 0} \sum_{k=1}^{n} (z_k - z_{k-1}) f(\zeta_k)$,

where $\|P\|$ = norm of the partition

$$= \max_k \{t_k - t_{k-1} : 1 \le k \le n\},$$

exists and is equal to J, which is independent of z_k and ζ_k, we say that f is integrable along the arc Γ and we write $J = \int_\Gamma f(z)\,dz$.

Thus by definition,

$$J = \int_\Gamma f(z)\,dz = \lim_{\|P\| \to 0} \sum_{k=1}^{n} (z_k - z_{k-1}) f(\zeta_k) \quad \cdots (A)$$

Since as $\|P\| \to 0$ $\max |z_k - z_{k-1}| \to 0$, the above can be also be written as

$$\int_\Gamma f(z)\,dz = \lim_{\max|z_k - z_{k-1}| \to 0} \sum_{k=1}^{n} (z_k - z_{k-1}) f(\zeta_k) \quad \cdots (B)$$

Sufficient condition for Integrability

~~Statement~~: If f is continuous on a rectifiable arc Γ, then f is integrable along Γ.

Example If Γ is a rectifiable arc joining the points a and b, prove that

(i) $\int_\Gamma dz = b - a$ (ii) $\int_\Gamma k\,dz = k(b-a)$ (iii) $\int_\Gamma z\,dz = \frac{1}{2}(b^2 - a^2)$

Solution (i) Here $f(z) = 1$ and so the integral exists since the integrand is continuous on Γ. We divide Γ into smaller arcs by the points $a = z_0, z_1, z_2, \ldots, z_{n-1}$; $z_n = b$ and form the sum

$S = \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})$, where ζ_k is a point on Γ between z_{k-1} and z_k

$= \sum^{n} (z_k - z_{k-1}) = b - a$

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Since $\int_\Gamma f(z)dz$ exists, it follows that

$$\int_\Gamma f(z)dz = \lim_{\max|z_k - z_{k-1}| \to 0} S = b-a.$$

(ii) Here $f(z) = k$ and so the integral exists since the integrand is continuous on Γ. We divide Γ into smaller arcs by the points $a = z_0, z_1, z_2, \dots, z_{n-1}, z_n = b$ and from the sum

$$S = \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}),$$ where ζ_k is a point on Γ between z_{k-1} ~~and~~ and z_k.

$$= \sum_{k=1}^{n} k(z_k - z_{k-1}) = k(b-a).$$

Since $\int_\Gamma f(z)dz$ exists, it follows that

$$\int_\Gamma f(z)dz = \lim_{\max|z_k - z_{k-1}| \to 0} S = k(b-a).$$

(iii) Here $f(z) = z$ and so the integral exists since the integrand is cnt. on Γ. Let AB be the given rectifiable arc Γ. Let a, b be the affixes of the points A and B. We divide Γ into smaller arcs by the points $a = z_0, z_1, \dots, z_{n-1}, z_n = b$ and from the sum

$$S = \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}),$$ where ζ_k is a point on Γ between z_{k-1} and z_k.

$$= \sum_{k=1}^{n} \zeta_k (z_k - z_{k-1}).$$

We take $\zeta_k = z_k$ and $\zeta_k = z_{k-1}$ and obtain sums

$$S_1 = \sum_{k=1}^{n} z_k (z_k - z_{k-1})$$

$$S_2 = \sum_{k=1}^{n} z_{k-1} (z_k - z_{k-1}).$$

Since $\int_\Gamma f(z)dz$ exists, $\lim_{\max|z_k - z_{k-1}| \to 0} S$, $\lim_{\max|z_k - z_{k-1}| \to 0} S_1$ and $\lim_{\max|z_k - z_{k-1}| \to 0} S_2$ all exist and to the same limit $\int_\Gamma f(z)dz$. Therefore,

$$2\int_\Gamma f(z)dz = 2\lim_{\max|z_k - z_{k-1}| \to 0} S = \lim_{\max|z_k - z_{k-1}| \to 0} (S_1 + S_2)$$

$$= \lim_{\max|z_k - z_{k-1}| \to 0} \sum_{k=1}^{n} (z_k + z_{k-1})(z_k - z_{k-1})$$

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$$\lim_{\max|z_k - z_{k-1}| \to 0} \sum_{k=1}^{n} (z_k^2 - z_{k-1}^2) = \lim_{\max|z_k - z_{k-1}| \to 0} (z_n^2 - z_0^2)$$

$$= b^2 - a^2$$

i.e., $\int_\Gamma z\,dz = \frac{1}{2}(b^2 - a^2)$.

Some elementary properties:

If f and g are continuous functions over a rectifiable arc Γ then

1. $\int_\Gamma \{f(z) \pm g(z)\}\,dz = \int_\Gamma f(z)\,dz \pm \int_\Gamma g(z)\,dz$
2. $\int_\Gamma k f(z)\,dz = k\int_\Gamma f(z)\,dz$, k being a constant.
3. If $-\Gamma$ denotes the arc Γ described in the opposite sense then $\int_{-\Gamma} f(z)\,dz = -\int_\Gamma f(z)\,dz$.
4. If Γ is the rectifiable arc consisting of a finite number of rectifiable arcs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ i.e., $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ then

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^{n} \int_{\Gamma_i} f(z)\,dz.$$

Integration along a regular arc

Let $f = u + iv$ be cont. on a regular arc Γ whose equation is $z = z(t) = x(t) + iy(t)$, $a \le t \le b$. On the arc Γ this becomes $f(z) = u(x(t), y(t)) + iv(x(t), y(t))$. Since $u(x(t), y(t))$, $v(x(t), y(t))$ are functions of t we can write $f(z) = \varphi(t) + i\psi(t) = F(t)$, where $z = x(t) + iy(t)$ is a point on Γ corresponding to the ~~parameter~~ t on $a \le t \le b$. It can be proved that f is integrable and

$$\int_\Gamma f(z)\,dz = \int_a^b \{\varphi(t) + i\psi(t)\}\{\dot{x}(t) + i\dot{y}(t)\}\,dt.$$

More generally, it can be shown that if $f(z)$ is continuous on a contour Γ it is integrable along Γ, the value of its integral being the sum of the integrals of $f(z)$ along the regular arcs of which Γ is composed.

<u>Note</u> The importance of the above result lies in the fact that it reduces the problem of evaluating complex integral to the integration of two real cont. functions of real variables.

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An inequality for complex integral

Theorem If f is cont. on a contour Γ of length L and if there exists a positive number M such that $|f(z)| \le M$ for all $z \in \Gamma$, then

$$\left| \int_\Gamma f(z)\,dz \right| \le ML \quad (ML\text{-formula}).$$

Proof In the first case we shall show that if $F(t) = U(t) + iV(t)$ be any continuous complex valued function of a real variable t in $a \le t \le b$, then

$$\left| \int_a^b F(t)\,dt \right| \le \int_a^b |F(t)|\,dt.$$

Let $P = \{a = t_0, t_1, \dots, t_{r-1}, t_r, \dots, t_n = b\}$ be any partition of $[a,b]$. From the definition of the integral of a continuous function of a real variable we get

$$\int_a^b U(t)\,dt = \lim_{\|P\| \to 0} \sum_{r=1}^{n} U(t_r)(t_r - t_{r-1}) \text{ and}$$

$$\int_a^b V(t)\,dt = \lim_{\|P\| \to 0} \sum_{r=1}^{n} V(t_r)(t_r - t_{r-1}).$$

Therefore,

$$\int_a^b \{U(t) + iV(t)\}\,dt = \lim_{\|P\| \to 0} \sum_{r=1}^{n} \{U(t_r) + iV(t_r)\} \times (t_r - t_{r-1}) \quad \text{------ (1)}$$

Now,

$$\left| \sum_{r=1}^{n} \{U(t_r) + iV(t_r)\}(t_r - t_{r-1}) \right|$$
$$\le \sum_{r=1}^{n} |U(t_r) + iV(t_r)|\,(t_r - t_{r-1}) \quad \text{----- (2).}$$

Since $U(t) + iV(t)$ is integrable, so is $|U(t)$ because both are continuous.

Therefore,

$$\int_a^b |U(t) + iV(t)|\,dt = \lim_{\|P\| \to 0} \sum_{r=1}^{n} |U(t_r) + \quad (t_r - t_{r-1}) \text{ ---}$$

By (1) $\lim\limits_{\|P\| \to 0} \sum\limits_{r=1}^{n} \{U(t_r) + iV(t_r)\}(t_r - t_{r-1}$

and so $\lim\limits_{\|P\| \to 0} \left| \sum\limits_{r=1}^{n} \{U(t_r) + iV(t_r)\}(t_r - t_{r-1}\right.$

$$= \left| \lim_{\|P\| \to 0} \sum_{r=1}^{n} \{U(t_r) + iV(t_r)\}(t_r - t_{r-1}) \right| \text{ --}$$

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From (2) we get

$$\lim_{\|P\|\to 0} \left| \sum_{r=1}^{n} \{U(t_r) + i V(t_r)\} (t_r - t_{r-1}) \right|$$

$$\leq \lim_{\|P\|\to 0} \sum_{r=1}^{n} \left| U(t_r) + i V(t_r) \right| (t_r - t_{r-1})$$

i.e.,

$$\left| \lim_{\|P\|\to 0} \sum_{r=1}^{n} \{U(t_r) + iV(t_r)\} (t_r - t_{r-1}) \right|$$

$$\leq \int_a^b |U(t) + iV(t)|\, dt \quad [\text{by (3) and (4)}]$$

i.e., $$\left| \int_a^b \{U(t) + iV(t)\} dt \right| \leq \int_a^b |U(t) + iV(t)|\, dt$$

i.e., $$\left| \int_a^b F(t) dt \right| \leq \int_a^b |F(t)| dt.$$

Now we show that

$$\left| \int_\Gamma f(z) dz \right| \leq ML.$$

Let $f(z) = u(x,y) + iv(x,y)$ and the equation of Γ be $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, where $\dot{x}(t)$, $\dot{y}(t)$ are continuous and do not vanish simultaneously. Now we have,

$$\int_\Gamma f(z) dz = \int_a^b \{U(t) + iV(t)\} \{\dot{x}(t) + i\dot{y}(t)\} dt$$

where $f(z) = U(t) + i V(t)$ on Γ because on Γ $u(x,y) = u(x(t), y(t)) = U(t)$ and $v(x,y) = v(x(t), y(t)) = V(t)$.

Therefore,

$$\left| \int_\Gamma f(z) dz \right| = \left| \int_a^b \{U(t) + iV(t)\} \{\dot{x}(t) + i\dot{y}(t)\} dt \right|$$

$$\leq \int_a^b |U(t) + iV(t)|\, |\dot{x}(t) + i\dot{y}(t)|\, dt$$

$$= \int_a^b |f(z)|\, |\dot{z}(t)|\, dt$$

$$\leq M \int_a^b |\dot{z}(t)| dt = ML, \quad \text{because } L = \int_a^b |\dot{z}(t)|\, dt$$

This completes the proof.

Note: The ML-formula is also valid for a noncontinuous integrable function.

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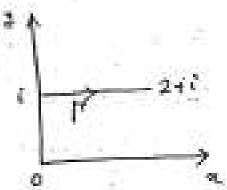
Example Let $f(z) = \frac{1}{z^2}$ and Γ be the straight line joining the points i and $2+i$. Show that
$\left|\int_\Gamma f(z)\,dz\right| \leq 2$.

Soln Here L = length of the contour $= |2+i-i| = 2$ and on Γ
$|f(z)| = \frac{1}{|z|^2} = \frac{1}{x^2+y^2} = \frac{1}{x^2+1} \leq 1 = M$ (say), because on Γ, $y=1$ and $x \geq 0$. Hence by ML-formula we get $\left|\int f(z)\,dz\right| \leq ML = 2$



Example Evaluate $\int_C \frac{dz}{(z-\alpha)^n}$ where n is any integer positive, negative or zero and C denotes the circle $|z-\alpha| = r$, described in the positive sense.

Soln The parametric equation of the circle is $z-\alpha = re^{i\theta}$, $0 \leq \theta \leq 2\pi$. Then $dz = ire^{i\theta}d\theta$ and so,

$$\int_C \frac{dz}{(z-\alpha)^n} = \int_0^{2\pi} \frac{ire^{i\theta}\,d\theta}{r^n e^{in\theta}} = \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta}\,d\theta.$$

If $n=1$, $\int_\Gamma \frac{dz}{(z-\alpha)^n} = i\int_0^{2\pi} d\theta = 2\pi i$.

If $n \neq 1$,

$$\int_\Gamma \frac{dz}{(z-\alpha)^n} = \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta}\,d\theta$$

$$= \frac{i}{r^{n-1}} \left[\frac{e^{i(1-n)\theta}}{i(1-n)} \right]_0^{2\pi}$$

$$= \frac{1}{(1-n)r^{n-1}} \left[e^{2\pi i(1-n)} - 1 \right] = 0.$$

Ex Evaluate $\int_C \bar{z}\,dz$ where C is the upper of the circle $|z| = 1$ from $z=-1$ to $z=1$.

Soln The parametric equation of the upper half of the circle $|z| = 1$ from $z=-1$ to $z=1$ is z [illegible] θ varies from π to 0.) So,

$$\int_C \bar{z}\,dz = \int_\pi^0 e^{-i\theta}\, ie^{i\theta}\,d\theta = i\int_\pi^0 d\theta = i\,[\theta]_\pi^0 = i\,[0-\pi$$

Ex Evaluate $\int_C (\bar{z}^2 dz + z^2 d\bar{z})$ along the curve defined by $z^2 + 2z\bar{z} + \bar{z}^2 = (2-2i)z + (2+2i)\bar{z}$ from point $z=1$ to $z = 2+2i$.

Soln Putting $z = x+iy$, $\bar{z} = x-iy$ the equation of curve C becomes $\bigcirc$

$(x+iy)^2 + 2(x+iy)(x-iy) + (x-iy)^2$

$= (2-2i)(x+iy) + (2+2i)(x-iy)$

i.e., $x^2-y^2+2ixy + 2(x^2+y^2) + x^2-y^2-2ixy$

$= 2x+2iy-2ix+2y+2x+2ix-2iy+2y$

i.e. $4x^2 = 4(x+y)$

i.e. $x^2 = x+y$.

Also the integrand becomes

$\bar{z}^2 dz + z^2 d\bar{z} = (x-iy)^2(dx+idy) + (x+iy)^2(dx-idy)$

$= (x^2-y^2)dx + 2xy\,dy + i\{(x^2-y^2)dy - 2xy\,dx\}$

$+ (x^2-y^2)dx + 2xy\,dy - i\{(x^2-y^2)dy - 2xy\,dx\}$

$= 2(x^2-y^2)dx + 4xy\,dy$

So, on the curve $C: x^2 = x+y$ it becomes

$\bar{z}^2 dz + z^2 d\bar{z} = 2\{x^2-(x^2-x)^2\}dx + 4x(x^2-x)\,dy$

$= 2\{x^2-x^4-x^2+2x^3\}dx + 4(x^2-x)(2x-1)\,dx$

$= (6x^4-8x^3+4x^2)\,dx$

For $z=1$ we get $x=1$ and for $z=2+2i$ we get $x=2$. Therefore,

$\int_C \bar{z}^2 dz + z^2 d\bar{z} = \int_{x=1}^{2}(6x^4-8x^3+4x^2)\,dx$

$= \left[\frac{6x^5}{5} - \frac{8x^4}{4} + \frac{4x^3}{3}\right]_1^2$

$= \frac{6\times 32}{5} - 2\times 16 + \frac{4\times 8}{3} - \frac{6}{5} + 2 - \frac{4}{3} = \frac{248}{15}$.

Cauchy's Fundamental Theorem (Cauchy – Goursat Theorem)

It is one of the most important results in the theory of functions of a complex variable. It has very far reaching implication in a sense almost every thing to follow will depend in one way or another on it.

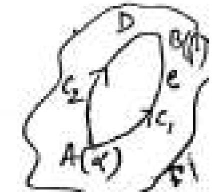
Statement: If f is analytic within and on a simple closed rectifiable curve C then $\int_C f(z)\,dz = 0$.

We shall denote the integration in the positive sense by the symbol $\oint f(z)\,dz$ or simply by $\int_C f(z)\,dz$ and an integration in the negative sense by the symbol $\oint_C f(z)\,dz$ or simply by $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$.

Consequences of Cauchy's Fundamental Theorem

I. Let f be analytic in a simply connected region R and let α and β be any two points in R. Then $\int_\alpha^\beta f(z)\,dz$ is independent of the path in R joining α and β.

Proof Let us join the points $A(\alpha)$ and $B(\beta)$ by two curves C_1 and C_2. Then we get a simple closed rectifiable curve $ACBDA$.



Then because f is analytic in R, by Cauchy's Fundamental Theorem we get

$$\int_{ACBDA} f(z)\,dz = 0. \text{ i.e., } \int_{ACB} f(z)\,dz + \int_{BDA} f(z)\,dz = 0.$$

i.e., $\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$

i.e., $\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$. i.e., $\int_{C_1} f(z)\,dz = \int_{C_2}$

So, $\int_\alpha^\beta f(z)\,dz$ is independent of the path in α and β.

II Let C_1 and C_2 be two simple closed cu[rves] C_2 lying wholly within C_1. If f is analytic [on] the closed annulus determined by C_1 an[d C_2] then $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$.

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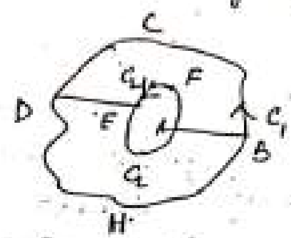
Proof: We introduce two cuts AB and DE joining C_1 and C_2. Since f is analytic in the annular region determined by C_1 and C_2, we get



$$\int_{ABCDEFA} f(z)\,dz = 0 \quad \text{and} \quad \int_{EDHBAGE} f(z)\,dz = 0$$

i.e., $$\int_{AB} f(z)\,dz + \int_{BCD} f(z)\,dz + \int_{DE} f(z)\,dz + \int_{EFA} f(z)\,dz = 0 \quad ---(1)$$

and $$\int_{ED} f(z)\,dz + \int_{DHB} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{AGE} f(z)\,dz = 0$$

i.e., $$-\int_{DE} f(z)\,dz + \int_{DHB} f(z)\,dz + -\int_{AB} f(z)\,dz + \int_{AGE} f(z)\,dz = 0 \quad ---(2)$$

Adding (1) and (2) we get

$$\left\{ \int_{BCD} f(z)\,dz + \int_{DHB} f(z)\,dz \right\} + \left\{ \int_{EFA} f(z)\,dz + \int_{AGE} f(z)\,dz \right\} = 0$$

i.e., $$\int_{BCDHB} f(z)\,dz + \int_{EFAGE} f(z)\,dz = 0$$

i.e., $$\oint_{C_1} f(z)\,dz + \oint_{-C_2} f(z)\,dz = 0.$$

i.e., $$\oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz = 0.$$

i.e., $$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$$

III If f is analytic in the annulus determined by C_1 and C_2 and if C_3 is any simple closed curve lying wholly within C_1 and containing within it C_2 then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz = \oint_{C_3} f(z)\,dz.$$

Note: It follows that the value of the integral of an analytic function f is unaltered by the deformation of the contour provided the contour ~~in~~ crosses no singularity of f during the process of deformation.

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