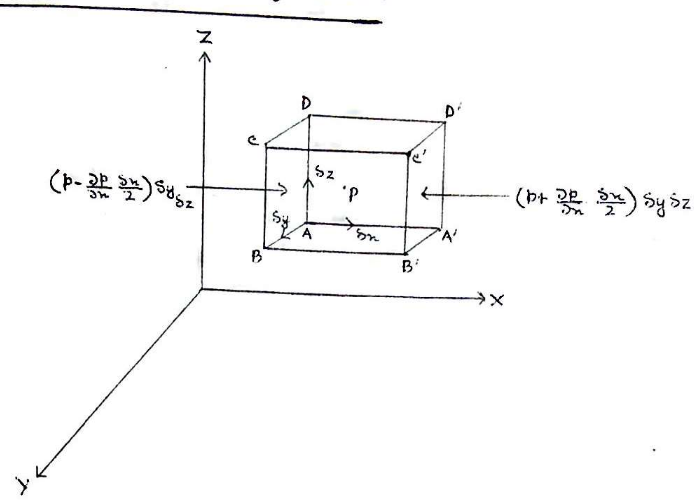
- Euler's dynamical equation of motion :-



Let p be the pressure and ρ be the density at a point $P(x, y, z)$ in a inviscid fluid. Consider an elementary parallelopiped with edges of lengths $\delta x, \delta y, \delta z$ parallel to their respective co-ordinate axes having P at its centre as shown in the above figure. Let (u, v, w) be the componants of velocity and (X, Y, Z) be the componants of external force per unit mass at time t at P. Then if $p = f(x, y, z)$ we have,

force on the plane through P & parallel to $ABCD = p\,\delta y\,\delta z$

$\therefore$ Force on the face $ABCD = f(x - \frac{1}{2}\delta x, y, z)\,\delta y\,\delta z$

$= \{f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \cdots\}\,\delta y\,\delta z$ (expanding by Taylor's theorem)

Again, the force on the face $A'B'C'D' = f(x + \frac{1}{2}\delta x, y, z)\,\delta y\,\delta z$

$= \{f + \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \cdots\}\,\delta y\,\delta z$

The net force along x-direction due to forces on the face $ABCD$ and $A'B'C'D'$

$= \{f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \cdots\}\,\delta y\,\delta z - \{f + \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \cdots\}\,\delta y\,\delta z$

$= -\frac{\partial f}{\partial x}\,\delta x\,\delta y\,\delta z$ (taking 1st order approximation)

$= -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$

The mass of the element is $\rho\,\delta x\,\delta y\,\delta z$

Hence the external force on the element in x-direction is $X\rho\,\delta x\,\delta y\,\delta z$.

Also we have $\frac{Du}{Dt}$ is the total acceleration of the element in x-direction.

By Newton 2nd law of motion in x-direction is given by

mass × (acceleration in x-direction) = sum of the components of external forces in x-direction

$$\therefore \rho\,\delta x\,\delta y\,\delta z\,\frac{Du}{Dt} = X\rho\,\delta x\,\delta y\,\delta z - \frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$

$$\Rightarrow \rho\frac{Du}{Dt} = X\rho - \frac{\partial p}{\partial x}$$

$$\Rightarrow \frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{---- (i)}$$

similarly the equation of motion in y and z-directions are --

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{--- (ii)}$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \quad \text{--- (iii)}$$

Rewriting (i) (ii) and (iii) we get the Euler's dynamical equation of motion in cartesian co-ordinates are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{---- (iv)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{--- (v)}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \quad \text{---- (vi)}$$

Now $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

$\vec{F} = X\hat{i} + Y\hat{j} + Z\hat{k}$

Then since $\vec{\nabla}p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$

(i), (ii) and (iii) may be combined in vector form as ----

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p \quad \text{--- (vii)}$$

which is Euler's equation of motion in vector form.

Equation (vii) can be written ~~as~~ as --

$$\frac{\partial\vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p \quad \text{-- (viii)}$$

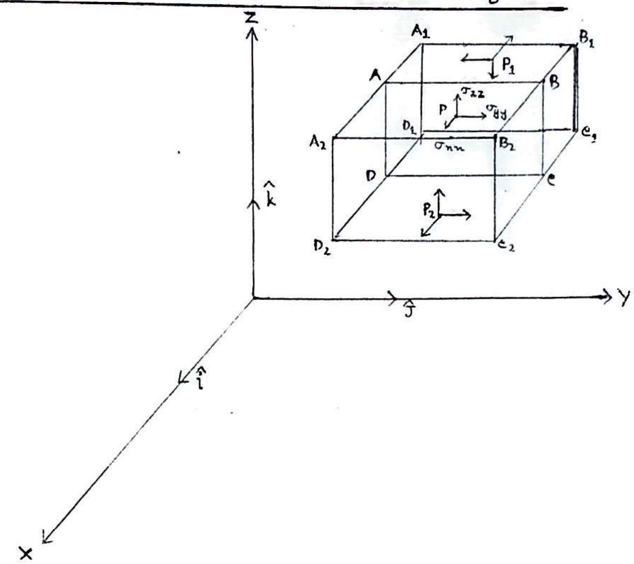
Again $\vec{\nabla}(\vec{q}\cdot\vec{q}) = 2\left[\vec{q}\times \operatorname{curl}\vec{q} + (\vec{q}\cdot\vec{\nabla})\vec{q}\right]$

$\Rightarrow \quad (\vec{q}\,\vec{\nabla})\vec{q} = \frac{1}{2}\vec{\nabla}(\vec{q}^{\,2}) - \vec{q}\times \text{curl}\,\vec{q} \quad \text{---(ix)}$

using (ix), (viii) becomes ---

$$\frac{\partial \vec{q}}{\partial t} - \vec{q}\times \text{curl}\,\vec{q} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p - \frac{1}{2}\vec{\nabla}(\vec{q}^{\,2})$$

- **The Navier-Stokes equation of a viscous fluid :-**



With $P(x,y,z)$ as centre and edges of lengths $\delta x, \delta y\ \delta z$ parallel to the fixed co-ordinate axes we have constructed an elementary rectangular parallelopiped as shown in the above figure. We consider the motion of above mentioned parallelopiped of viscous fluid. We suppose that the element is moving with the fluid and the mass $\rho\,\delta x\,\delta y\,\delta z$ of the fluid remains constant where ρ is the density of the fluid. Let the co-ordinates of the points P_1 and P_2 be $(x-\frac{1}{2}\delta x, y, z)$ and $(x+\frac{1}{2}\delta x, y, z)$ respectively.

At P, the force components parallel to OX, OY, OZ on the rectangular surface ABCD of area $\delta y\,\delta z$ through P and having $\hat{i}$ as unit normal are $[\sigma_{xx}\,\delta y\,\delta z,\ \sigma_{xy}\,\delta y\,\delta z,\ \sigma_{xz}\,\delta y\,\delta z]$.

At P_2, since $\hat{i}$ is the unit normal measured outwards from the fluid the corresponding force components on the rectangular surface $A_2B_2C_2D_2$ of area $\delta y\,\delta z$ are ---

$$\left[\left(\sigma_{xx}+\frac{1}{2}\delta x\frac{\partial \sigma_{xx}}{\partial x}\right)\delta y\,\delta z,\ \left(\sigma_{xy}+\frac{1}{2}\delta x\frac{\partial \sigma_{xy}}{\partial x}\right)\delta y\,\delta z,\ \left(\sigma_{xz}+\frac{1}{2}\delta x\frac{\partial \sigma_{xz}}{\partial x}\right)\delta y\,\delta z\right]$$

At P_1 since $-\hat{i}$ is the unit normal measured outwards from the fluid, the corresponding force components on the rectangular surface $A_1 B_1 C_1 D_1$ of area $\delta y\, \delta z$ are $\left[-\left(\sigma_{xx} - \frac{1}{2}\delta x \frac{\partial \sigma_{xx}}{\partial x}\right)\delta y\, \delta z,\ -\left(\sigma_{xy} - \frac{1}{2}\delta x \frac{\partial \sigma_{xy}}{\partial x}\right)\delta y\, \delta z\right.$

$$\left.-\left(\sigma_{xz} - \frac{1}{2}\delta x \frac{\partial \sigma_{xz}}{\partial x}\right)\delta y\, \delta z\right] \quad \text{--- (ii)}$$

Hence the forces on the parallel planes $A_2 B_2 C_2 D_2$ and $A_1 B_1 C_1 D_1$ passing through the points P_1 and P_2 are equivalent to a single force at P with components $\left[\frac{\partial \sigma_{xx}}{\partial x}\delta x\, \delta y\, \delta z,\ \frac{\partial \sigma_{xy}}{\partial x}\delta x\, \delta y\, \delta z,\ \frac{\partial \sigma_{xz}}{\partial x}\delta x\, \delta y\, \delta z\right]$ --- (iii)

Together with couples whose moments are $-\sigma_{xz}\, \delta x\, \delta y\, \delta z$ about OY and $\sigma_{xy}\, \delta x\, \delta y\, \delta z$ about OZ --- (iv)

Similarly the forces on the parallel planes perpendicular to the y-axis are equivalant to a single force at P with components $\left[\frac{\partial \sigma_{yx}}{\partial y}\delta x\, \delta y\, \delta z,\right.$

$\left.\frac{\partial \sigma_{yy}}{\partial y}\delta x\, \delta y\, \delta z,\ \frac{\partial \sigma_{yz}}{\partial y}\delta x\, \delta y\, \delta z\right]$ --- (v)

together with couples whose moments are $-\sigma_{yx}\, \delta x\, \delta y\, \delta z$ about OZ and $\sigma_{yz}\, \delta x\, \delta y\, \delta z$ about OX --- (vi)

Again the forces on the parallel planes perpendicular to the z-axis are equivalant to a single force at P with components $\left[\frac{\partial \sigma_{zx}}{\partial z}\delta x\, \delta y\, \delta z,\right.$

$\left.\frac{\partial \sigma_{zy}}{\partial z}\delta x\, \delta y\, \delta z,\ \frac{\partial \sigma_{zz}}{\partial z}\delta x\, \delta y\, \delta z\right]$ --- (vii)

Together with couples whose moments are ---- $-\sigma_{zy}\, \delta x\, \delta y\, \delta z$ about OX and $\sigma_{zx}\, \delta x\, \delta y\, \delta z$ about OY ---- (viii)

Thus the surface forces on all the six faces of the rectangular parallelopiped are equivalant to a single force at P having componants --

$$\left[\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\, \delta y\, \delta z,\ \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}\right)\delta x\, \delta y\, \delta z,\right.$$

$$\left.\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\delta x\, \delta y\, \delta z\right] \quad \text{--- (ix)}$$

together with a vector couple having components --

$$\left[(\sigma_{yz} - \sigma_{zy})\delta x\, \delta y\, \delta z,\ (\sigma_{zx} - \sigma_{xz})\delta x\, \delta y\, \delta z,\ (\sigma_{xy} - \sigma_{yx})\delta x\, \delta y\, \delta z\right]$$

-----(x)

Let $$\left.\begin{aligned} \vec{q} &= u\hat{i} + v\hat{j} + w\hat{k} \\ \vec{B} &= B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \end{aligned}\right\} \quad \text{(xi)}$$

be the velocity of the fluid at $P(x, y, z)$ at any time t and extended body force at P per unit mass respectively.

Clearly, the total body force of the elementary rectangular parallelopiped has components --- $(B_x \rho\, \delta x\, \delta y\, \delta z,\ B_y \rho\, \delta x\, \delta y\, \delta z,\ B_z \rho\, \delta x\, \delta y\, \delta z)$ ..... (xii)

Taking account of surface and body forces we find the total force component along the x-direction on the element of the fluid is given by ---

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\, \delta y\, \delta z + B_x \rho\, \delta x\, \delta y\, \delta z$$

The total acceleration along the x-axis is. $\frac{Du}{Dt}$

$\therefore$ By Newton's 2nd law of motion the equation of motion in x-direction is

mass x acceleration in x-direction = sum of the components of surface and body forces along x-direction

$$\Rightarrow \rho \frac{Du}{Dt}\delta x\, \delta y\, \delta z = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\, \delta y\, \delta z + B_x \rho\, \delta x\, \delta y\, \delta z$$

$$\Rightarrow \rho \frac{Du}{Dt} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + B_x \rho \quad \text{--- (xiii)}$$

similarly, the equation of motion in y and z direction is.

$$\left.\begin{aligned} \rho \frac{Dv}{Dt} &= \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + B_y \rho \\ \rho \frac{Dw}{Dt} &= \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z \rho \end{aligned}\right\} \text{---- (xiv)}$$

The constitutive equation for a Newtonian viscous compressible fluid are given by

$$\sigma_{xx} = 2\mu \frac{\partial u}{\partial x} - \frac{2\mu}{3}\vec{\nabla}\cdot\vec{q} - p \qquad \sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\sigma_{yy} = 2\mu \frac{\partial v}{\partial y} - \frac{2\mu}{3}\vec{\nabla}\cdot\vec{q} - p \qquad \sigma_{yz} = \sigma_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\sigma_{zz} = 2\mu \frac{\partial w}{\partial z} - \frac{2\mu}{3}\vec{\nabla}\cdot\vec{q} - p \qquad \sigma_{zx} = \sigma_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

(xv)

using (xv) the equation (xiii) and (xiv) can be written as...

$$\rho \frac{Du}{Dt} = \rho B_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left\{2\frac{\partial u}{\partial x} - \frac{2}{3}(\vec{\nabla}.\vec{q})\right\}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$

$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$\rho \frac{Dv}{Dt} = \rho B_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left\{2\frac{\partial v}{\partial y} - \frac{2}{3}(\vec{\nabla}.\vec{q})\right\}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$

$$\rho \frac{Dw}{Dt} = \rho B_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left\{2\frac{\partial w}{\partial z} - \frac{2}{3}(\vec{\nabla}.\vec{q})\right\}\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \qquad \text{----- (xvi)}$$

The above three equations are called the Navier-Stokes equation of motion for a viscous compressible fluid in cartesian co-ordinate.

Let the co-efficients of viscosity μ be constant, then equation (xvi) may be expressed as —

$$\rho\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\vec{\nabla})\vec{q}\right] = \rho\vec{B} - \vec{\nabla}p + \mu\vec{\nabla}^2 q + \frac{\mu}{3}\vec{\nabla}(\vec{\nabla}.\vec{q}) \quad \text{... (xvii)}$$

Now, $\vec{q}\times(\vec{\nabla}\times\vec{q}) = \frac{1}{2}\vec{\nabla}(\vec{q}.\vec{q}) - (\vec{q}.\vec{\nabla})\vec{q}$

$$= \vec{\nabla}(q^2/2) - (\vec{q}.\vec{\nabla})\vec{q}$$

$$\Rightarrow (\vec{q}.\vec{\nabla})\vec{q} = \vec{\nabla}(q^2/2) - \vec{q}\times(\vec{\nabla}\times\vec{q})$$

Again $\vec{\nabla}\times(\vec{\nabla}\times\vec{q}) = \vec{\nabla}(\vec{\nabla}.\vec{q}) - (\vec{\nabla}.\vec{\nabla})\vec{q}$

$$= \vec{\nabla}(\vec{\nabla}.\vec{q}) - \vec{\nabla}^2\vec{q}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}.\vec{q}) = \vec{\nabla}\times(\vec{\nabla}\times\vec{q}) + \vec{\nabla}^2\vec{q}$$

using this (xvii) may be written as ....

$$\rho\left[\frac{\partial \vec{q}}{\partial t} + \vec{\nabla}(q^2/2) - \cancel{(\vec{q}.\vec{\nabla})}\,\vec{q}\times(\vec{\nabla}\times\vec{q})\right] = \rho\vec{B} - \vec{\nabla}p + \frac{4\mu}{3}\vec{\nabla}(\vec{\nabla}.\vec{q})$$

$$- \mu\vec{\nabla}\times(\vec{\nabla}\times\vec{q}) \quad \text{.... (xviii)}$$

Let ρ and μ be constants for the given incompressible fluid. Further for a such fluid $\vec{\nabla}\cdot\vec{q} = 0$ if $\nu = \frac{\mu}{\rho}$ be the kinematic viscosity then from (XVIII) the Navier - Stokes equation becomes —

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\,\vec{\nabla})\vec{q} = \vec{B} - \frac{1}{\rho}\vec{\nabla}p + \nu\,\vec{\nabla}^2 q$$

For a non-viscous incompressible fluid, we get $\mu = 0$ and the above equation becomes —

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\vec{\nabla})\vec{q} = \vec{B} - \frac{1}{\rho}\vec{\nabla}p$$

which coincides the Euler's dynamical equation of motion.

- **Flow and Circulation :-**

If A and P be any points in a fluid, then the value of the integral

$$\int_A^P \vec{q}\cdot d\vec{r} \quad \text{where } \vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= \int_A^P u\,dx + v\,dy + w\,dz$$

taken any path from A to P is called the flow along the path from A to P.

When a velocity potential ϕ exists, the flow from A to P equal to

$$= \int_A^P u\,dx + v\,dy + w\,dz$$

$$= \int_A^P -\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= -\int_A^P d\phi = \phi_A - \phi_P$$

The flow round a closed curve is known as the circulation round the curve.

Let C be the closed curve and Γ be the circulation, then we have —

$$\Gamma = \int_C u\,dx + v\,dy + w\,dz = \int_C \vec{q}\cdot d\vec{r}$$

where the line integral is taken round C in a counter clockwise direction.

Kelvin's minimum energy theorem :-

The irrotational motion of a liquid occupying a simply connected region has less kinetic energy than any other motion consistent with the same normal ~~rate~~ velocity of the boundary.

$\Rightarrow$ Let T_1 be the kinetic energy $\vec{q_1}$ be the fluid velocity of the actual irrotational motion with a velocity potential ϕ, then

$$\vec{q_1} = -\vec{\nabla}\phi \quad \text{--- (i)}$$

Let T_2 be the kinetic energy, $\vec{q_2}$ be the fluid velocity of any other possible state of motion consistent with the same normal velocity of the boundary S.

The continuity equations for the above two motions are :-

$$\vec{\nabla}.\vec{q_1} = 0 \quad \text{and} \quad \vec{\nabla}.\vec{q_2} = 0 \quad \text{--- (ii)}$$

Let $\hat{n}$ ~~be~~ denote the unit normal at a point of S. Then using the fact that the boundary has the same normal ~~rate~~ velocity in both motions, we have

$$\hat{n}.\vec{q_1} = \hat{n}.\vec{q_2} \quad \text{--- (iii)}$$

Now $T_1 = \frac{1}{2}\rho\int_V q_1^2\, dV = \frac{1}{2}\rho\int_V \vec{q_1}^2\, dV$.

$T_2 = \frac{1}{2}\rho\int_V q_2^2\, dV = \frac{1}{2}\rho\int_V \vec{q_2}^2\, dV$

$$\therefore T_2 - T_1 = \frac{1}{2}\rho\int_V (\vec{q_2}^2 - \vec{q_1}^2)\, dV$$

$$= \frac{1}{2}\rho\int_V \{2\vec{q_1}.(\vec{q_2}-\vec{q_1}) + (\vec{q_2}-\vec{q_1})^2\}\, dV$$

$$= \rho\int_V \vec{q_1}(\vec{q_2}-\vec{q_1})\, dV + \frac{1}{2}\rho\int_V (\vec{q_2}-\vec{q_1})^2\, dV$$

$$= -\rho\int_V (\vec{\nabla}\phi)(\vec{q_2}-\vec{q_1})\, dv + \frac{1}{2}\rho\int_V (\vec{q_2}-\vec{q_1})^2\, dV$$

[using (i)]

But $\vec{\nabla}\cdot[\phi(\vec{q_2}-\vec{q_1})]$

$$= \phi[\vec{\nabla}\cdot(\vec{q_2}-\vec{q_1})] + (\vec{\nabla}\phi)(\vec{q_2}-\vec{q_1})$$

$$= (\vec{\nabla}\phi)(\vec{q_2}-\vec{q_1}) \quad \text{(using (ii))}$$

$$\therefore \int_V (\vec{\nabla}\phi)(\vec{q_2}-\vec{q_1})\, dV = \int_V \vec{\nabla}\cdot[\phi(\vec{q_2}-\vec{q_1})]$$

$$= \int_S \phi\, \hat{n}\cdot(\vec{q_2}-\vec{q_1})\, dS \quad \text{(by divergence theorem)}$$

$$= 0 \quad \text{(using (iii))}$$

Thus $T_2 - T_1 = \frac{1}{2}\rho \int_V (\vec{q_2}-\vec{q_1})^2\, dV \geqslant 0$

$$\Rightarrow T_2 \geqslant T_1$$

Hence the result.

$$\sigma_{x'x} = l_2 \sigma_{xx} + m_2 \sigma_{yx} + n_2 \sigma_{zx}$$

$$\sigma_{y'y'} = l_2 \sigma_{x'x} + m_2 \sigma_{x'y} + n_2 \sigma_{x'z}$$

$$= l_2 (l_2 \sigma_{xx} + m_2 \sigma_{yx} + n_2 \sigma_{zx})$$

$$+ m_2 (l_2 \sigma_{xy} + m_2 \sigma_{yy} + n_2 \sigma_{zy})$$

$$+ n_2 (l_2 \sigma_{xz} + m_2 \sigma_{yz} + n_2 \sigma_{zz})$$

$$= l_2 \sigma_{xx} + m_2^2 \sigma_{yy} + n_2^2 \sigma_{zz}$$

$$+ 2 l_2 m_2 \sigma_{xy}$$