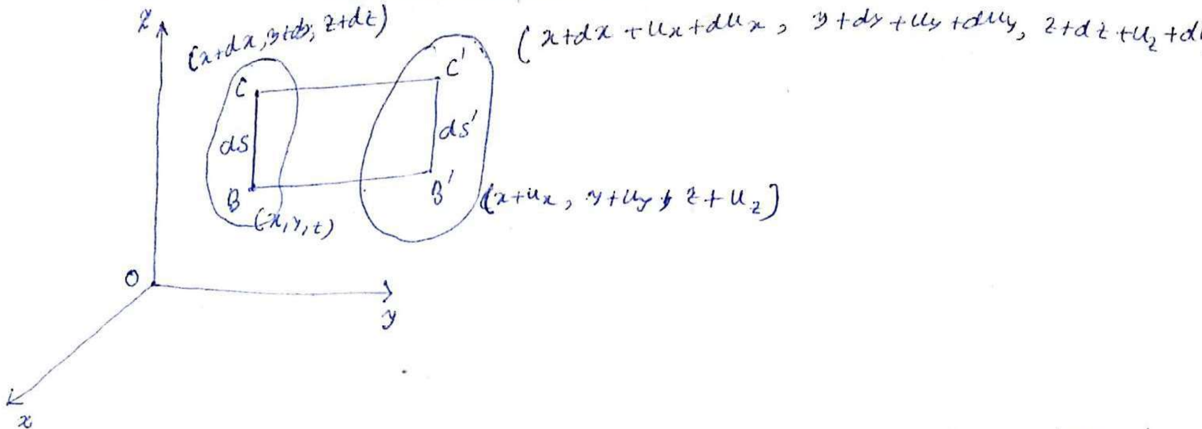


Let B and C be two infinitesimally close neighbouring points in a body before deformation and B' and C' are their new positions after deformation. The cartesian co-ordinates of these four points are as follows –

co-ordinates of B: (x, y, z)

co-ordinates of C: $(x+dx, y+dy, z+dz)$

co-ordinates of B': $(x+u_x, y+u_y, z+u_z)$

co-ordinates of C': $(x+dx+u_x+du_x, y+dy+u_y+du_y, z+dz+u_z+du_z)$

where u_x, u_y, u_z are the displacement components of (x, y, z) and u_x+du_x, u_y+du_y, u_z+du_z are the displacement components of $(x+dx, y+dy, z+dz)$. In the original configuration, the square of the distance between the points B and C is

$$ds^2 = dx^2 + dy^2 + dz^2$$

similarly, in the deformed state, $ds'^2 = (dx+du_x)^2 + (dy+du_y)^2 + (dz+du_z)^2$

Now, $dx + du_x = dx + \frac{\partial u_x}{\partial x}.dx + \frac{\partial u_x}{\partial y}.dy + \frac{\partial u_x}{\partial z}dz$

$$= \left(1+\frac{\partial u_x}{\partial x}\right)dx + \frac{\partial u_x}{\partial y}.dy + \frac{\partial u_x}{\partial z}.dz$$

similarly, $dy + du_y = dy + \frac{\partial u_y}{\partial x}.dx + \frac{\partial u_y}{\partial y}.dy + \frac{\partial u_y}{\partial z}.dz$

$$= \frac{\partial u_y}{\partial x}.dx + \left(1+\frac{\partial u_y}{\partial y}\right)dy + \frac{\partial u_y}{\partial z}.dz$$

and, $dz + du_z = dz + \frac{\partial u_z}{\partial x}.dx + \frac{\partial u_z}{\partial y}.dy + \frac{\partial u_z}{\partial z}.dz$

$$= \frac{\partial u_z}{\partial x}.dx + \frac{\partial u_z}{\partial y}.dy + \left(1+\frac{\partial u_z}{\partial z}\right)dz$$

Therefore the difference between the squares of the length elements may be written as →

$$ds'^2 - ds^2 = (dx + du_x)^2 + (dy + du_y)^2 + (dz + du_z)^2 - (dx^2 + dy^2 + dz^2)$$

$$= \left[\left(1 + \frac{\partial u_x}{\partial x}\right) dx + \frac{\partial u_x}{\partial y} \cdot dy + \frac{\partial u_x}{\partial z} \cdot dz\right]^2$$

$$+ \left[\frac{\partial u_y}{\partial x} dx + \left(1 + \frac{\partial u_y}{\partial y}\right) dy + \frac{\partial u_y}{\partial z} dz\right]^2$$

$$+ \left[\frac{\partial u_z}{\partial x} \cdot dx + \frac{\partial u_z}{\partial y} dy + \left(1 + \frac{\partial u_z}{\partial z}\right) dz\right]^2 - (dx^2 + dy^2 + dz^2)$$

$$= 2\left[E_{xx} dx^2 + E_{yy} dy^2 + E_{zz} dz^2 + 2E_{xy}\, dx\, dy + 2E_{zx}\, dx \cdot dz + 2E_{yz}\, dy \cdot dz\right] \quad \ldots (1)$$

where
$$E_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right]$$

$$E_{yy} = \frac{\partial u_y}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial y}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right]$$

$$E_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] \quad \ldots (2)$$

$$E_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{1}{2}\left[\frac{\partial u_x}{\partial x} \cdot \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \cdot \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial x} \cdot \frac{\partial u_z}{\partial y}\right]$$

$$E_{yz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) + \frac{1}{2}\left[\frac{\partial u_x}{\partial y} \cdot \frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial y} \cdot \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \cdot \frac{\partial u_z}{\partial z}\right]$$

$$E_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \frac{1}{2}\left[\frac{\partial u_x}{\partial z} \cdot \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial z} \cdot \frac{\partial u_y}{\partial x} + \frac{\partial u_z}{\partial z} \cdot \frac{\partial u_z}{\partial x}\right]$$

It is obvious that $E_{xy} = E_{yx}$, $E_{yz} = E_{zy}$, $E_{zx} = E_{xz}$

$ds'^2 - ds^2$ is the fundamental ~~result~~ measure of deformation.

So, we find that to determine the deformation ~~of~~ in a region in the neighbourhood of any point $B(x, y, z)$, it is required to know the quantities E_{xx}, E_{xy} etc. at that point. These are called the components of Lagrangian finite ~~stress~~ strain Tensor at (x, y, z).

An immediate consequence of eq^n (1) is that, $ds'^2 - ds^2 = 0$ for all arbitrary small dx, dy, dz implies $E_{ij} = 0$, $(i, j = x, y, z)$. On the other hand strain components $E_{ij} = 0$, $(i, j = x, y, z)$ imply that $ds'^2 - ds^2 = 0$. But a deformation in which the length of every line element remains unchanged is rigid body motion. Hence the necessary and sufficient condition that a deformation of a body be a rigid body motion is that all components of strain tensor E_{ij} be zero throughout the body.

Infinitesimal Strains (small deformation)

If the displacement gradients are so small that squares and products of partial derivatives of u_x, u_y, u_z are negligible then Lagrangian finite strain Tensor reduces to infinitesimal strain Tensor denoted by e_{ij} $(i,j = x,y,z)$. $[e_{ij}]$

394

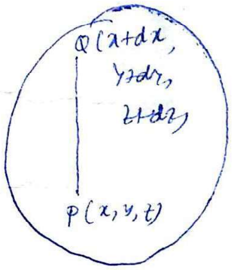
The components of infinitesimal strain Tensor are

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$e_{yy} = \frac{\partial u_y}{\partial y}, \quad e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$e_{zz} = \frac{\partial u_z}{\partial z}, \quad e_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)$$

Show that, every infinitesimal strain in a continuum is composed of a pure strain, a pure rotation, and translation. 391 390 '88



Let Q be an element within a small volume surrounding the point P.

Let (x,y,z) be the co-ordinates of P and $(x+dx, y+dy, z+dz)$ be the co-ordinates of Q in the undeformed state of the continuum. When the body undergoes deformation the components of displacement of the point P are taken to be u_x, u_y, u_z

So that P now occupies the position $(x+u_x, y+u_y, z+u_z)$. Let us assume that u_x, u_y, u_z together with their first derivatives are small quantities so that their squares and products can be neglected. The displacement components of Q are $(u_x+du_x, u_y+du_y, u_z+du_z)$. So we have

$$u_x + du_x = u_x + \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \frac{\partial u_x}{\partial z}dz$$

$$= u_x + \frac{\partial u_x}{\partial x}dx + \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)dz$$

$$- \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)dz$$

Similarly,

$$u_y + du_y = \cancel{u_y + \frac{\partial u_y}{}}\ u_y + \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)dx + \frac{\partial u_y}{\partial y}dy$$

$$+ \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)dz + \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)dx$$

$$- \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)dz$$

$$u_z + du_z = u_z + \frac{\partial u_z}{\partial x}.dx + \frac{\partial u_z}{\partial y}dy + \frac{\partial u_z}{\partial z}dz$$

$$= u_z + \frac{\partial u_z}{\partial z}dz + \frac{1}{2}\left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)dy$$

$$- \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)dy$$

USING Inifinitesimal strain components

i.e. $e_{xx} = \frac{\partial u_x}{\partial x}$, $e_{yy} = \frac{\partial u_y}{\partial y}$, $e_{zz} = \frac{\partial u_z}{\partial z}$

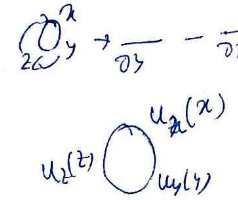
$e_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$, $e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$

$e_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)$

and putting $2\xi = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)$

$$2\eta = \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)$$

$$2\zeta = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$



$\therefore$ Displacement of Q consists of three parts

(i) u_x, u_y, u_z, which are same as the displacement compo-nent of P.

(ii) a displacement whose components are

$0.dx - \zeta.dy + \eta dz$.

$\zeta.dx + 0.dy - \xi dz$

$-\eta\, dx + \xi dy + 0.dz$

(iii) The displacement whose components are

$e_{xx}dx + e_{xy}.dy + e_{xz}.dz$

$e_{xy}.dx + e_{yy}\, dy + e_{yz}.dz$

$e_{xz}.dx + e_{yz}dy + e_{zz}dz$

In part (i) of the displacement Q is the same as the displacement of P and therefore equivalent to translatory motion of a rigid body.

In part (ii) is the displacement of Q relative to P which is equivalent to rigid body rotation through an angle ξ about an axis through P parallel to x-axis, rotation through an angle η about an axis through P parallel to y-axis and rotation through an angle ξ about an axis through P parallel to z-axis simultaneously.

~~In part (iii)~~ So part (i) and (ii) together constitute rigid body displacement of the point Q.

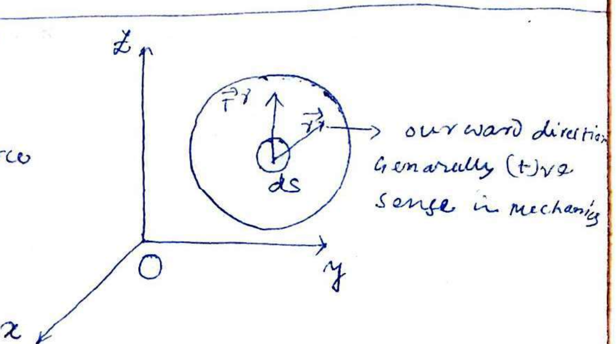
part (III) gives the deformation. If it is the displacement components of Q relative to P in the direction normal to the surface

$$\frac{1}{2}\{e_{xx}(dx)^2 + e_{yy}(dy)^2 + e_{zz}(dz)^2 + 2e_{yz}\,dy.dz + 2.e_{zx}\,dx\,dz + 2e_{xy}\,dx.dy\} = K^2$$

The quantities e_{xx}, e_{yy}, e_{zz} are called the longitudinal strain components and e_{xy}, e_{yz}, e_{zx} are the components of the shearing strain.

Mechanics of continuous media:

Due to gravitation there are two forces (i) surface forces (ii) Body force



Analysis of stress;

$$\vec{T}^r = T_{rx}\vec{i} + T_{ry}\vec{j} + T_{rz}\vec{k}$$
$$= \tau_{rx}\vec{i} + \tau_{ry}\vec{j} + \tau_{rz}\vec{k}$$
$$= \sigma_{rx}\vec{i} + \sigma_{ry}\vec{j} + \sigma_{rz}\vec{k}$$

$$\vec{T}^x = \tau_{xx}\vec{i} + \tau_{xy}\vec{j} + \tau_{xz}\vec{k}$$

$$\vec{T}^y = \tau_{yx}\vec{i} + \tau_{yy}\vec{j} + \tau_{yz}\vec{k}$$

$$\vec{T}^z = \tau_{zx}\vec{i} + \tau_{zy}\vec{j} + \tau_{zz}\vec{k}$$

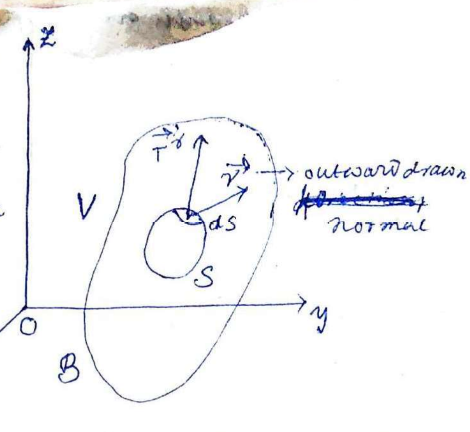
At atomic level a body is made up of discrete particles and its properties vary widely between neighbouring crystals. For, general particles it it is convenient to ignore molecular nature of matter and hence the discontinuity and variation at the microscopic level and instead of treat the matter, whether solids, liquid or gas as a continuum medium.

The study of the behaviour of such a continuum medium in motion or in equilibrium, under the action of externally applied forces is continuum mechanics.

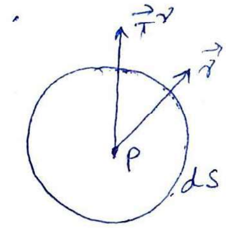
Forces: consider a continuous medium B occupying a region V at some time. Imagine a closed surface S within B.

"Forces acting on the material within the surface S are of two types

(i) Forces which act on all elements of the volume such as gravity. These are expressed as force per unit mass and are called 'body forces'.



(ii) Another type of force is the force exerted by material outside S on the material inside S across the boundary surface. These are called surface forces and are measured as forces/unit area of the surface.



Let ds be a small element of area surrounding the point P of the surface.

Let us draw the normal $\vec{\gamma}$ at P to the surface element ds in a specified sense as so integral. The component of the surface force across ds which the material on the side of ds towards which normal $\vec{\gamma}$ is drawn exerts on the material, on the other side are expressed by $\tau_{\gamma x} ds$, $\tau_{\gamma y} ds$, $\tau_{\gamma z} ds$. If we draw $\vec{\gamma}$ in the direction of x-axis, the components of surface force are $\tau_{xx} ds$, $\tau_{xy} ds$, $\tau_{xz} ds$. If $\vec{\gamma}$ be drawn in the direction of y-axis the components are $\tau_{yx} ds$, $\tau_{yy} ds$, $\tau_{yz} ds$ and if $\vec{\gamma}$ be in the direction of z-axis the components are $\tau_{zx} ds$, $\tau_{zy} ds$, $\tau_{zz} ds$. If ds be equal to unity then surface forces exerted by material on the side towards which normal $\vec{\gamma}$ is drawn on the material on the other side has components $\tau_{\gamma x}$, $\tau_{\gamma y}$, $\tau_{\gamma z}$. These are called the components of stress at P and the vector $\vec{T}$ of which these are the components is called the <u>stress vector</u> or ______ at P corresponding to the unit area at P whose normal is in the direction of $\vec{\gamma}$, therefore we can write

$$\vec{T}^{\gamma} = \tau_{\gamma x}\vec{i} + \tau_{\gamma y}\vec{j} + \tau_{\gamma z}\vec{k} \quad \text{---(1)}$$

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors in the direction of x, y, z.

Thus taking $\vec{\gamma}$ in the direction of x, y, z axes respectively

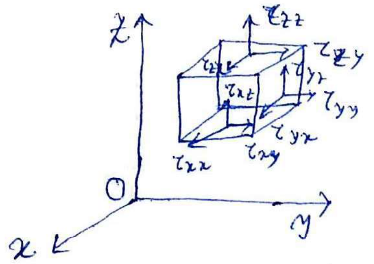
$$\left.\begin{aligned} \vec{T}^{x} &= \tau_{xx}\vec{i} + \tau_{xy}\vec{j} + \tau_{xz}\vec{k} \\ \vec{T}^{y} &= \tau_{yx}\vec{i} + \tau_{yy}\vec{j} + \tau_{yz}\vec{k} \\ \vec{T}^{z} &= \tau_{zx}\vec{i} + \tau_{zy}\vec{j} + \tau_{zz}\vec{k} \end{aligned}\right\} \cdots (2)$$

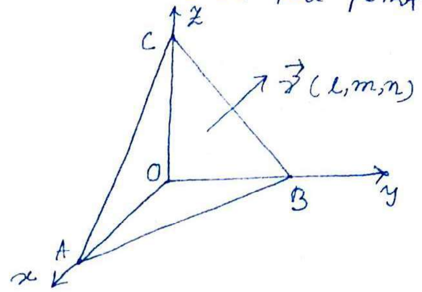
993

The nine stress vector component on the right hand side of (2) are the components of a second order cartesian tensor known as the <u>stress tensor</u>.

Matrix representation of the stress tensor is $\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$

The components $\tau_{xx}, \tau_{yy}, \tau_{zz}$ perpendicular to the planes, are called <u>normal stresses</u> and those acting tangential to the plane, i.e. $\tau_{xy}, \tau_{yx}, \tau_{yz}, \tau_{zy}, \tau_{xz}, \tau_{zx}$ are called the <u>shear stresses</u>. These nine components of stress tensor serve to determine complete the state of stress at the point O.





<u>Stress Tensor - Stress Vector relationship:</u> 2.2

Let us consider a tetrahedron of the continuum with one corner at the point 'O' and edges OA, OB, OC parallel to the co-ordinate axes and of infinitesimal lengths and face ABC perpendicular to the direction $\vec{\gamma}$ whose direction cosines are l, m, n and this direction is drawn outwards. Let ds_1, ds_2, ds_3 and ds be the areas of the faces OBC, OCA, OAB and ABC respectively.

$\therefore ds_1 = l.ds \quad , ds_2 = m.ds \quad , ds_3 = n.ds$

Let dv be the volume of the tetrahedron. Let f_x, f_y, f_z be the components of acceleration of the continuum in the direction of x, y, z axes respectively and $\rho X, \rho Y, \rho Z$ are the components of body force per unit volume of the body. Then the x-component of equation of motion of the material within the tetrahedron is

$$\rho\, dv\, f_x = \rho X dv - \tau_{xx} ds_1 - \tau_{yx} ds_2 - \tau_{zx} ds_3 + \tau_{\gamma x} ds$$

$$\text{or } \rho\, dv\, f_x = \rho X dv + (\tau_{\gamma x} - l\tau_{xx} - m\tau_{yx} - n\tau_{zx})\, ds$$

$$\text{or, } \rho \frac{dv}{ds} f_x = \rho X \frac{dv}{ds} + \tau_{\gamma x} - l\tau_{xx} - m\tau_{yx} - n\tau_{zx}$$

We make the dimension of the tetrahedron tends to zero but in such a manner that face ABC remains parallel to itself i.e. direction of $\vec{\gamma}$ remaining fixed. In the limit face ABC tends to pass through O therefore $\tau_{\gamma x}$ can be taken to be the value of the corresponding quantities of stress at the point O. Since the linear dimension of the tetrahedron $\frac{dv}{ds}$ tends to zero so we have

$$\tau_{\gamma x} = l\tau_{xx} + m\tau_{yx} + n\tau_{zx}$$

similarly,

$$\tau_{\gamma y} = l\tau_{xy} + m\tau_{yy} + n\tau_{zy}$$

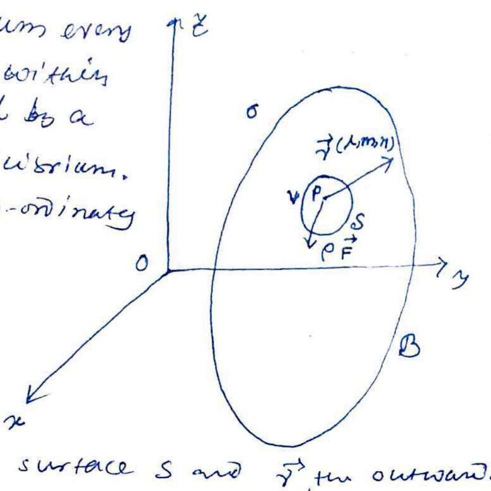
$$\tau_{\gamma z} = l\tau_{xz} + m\tau_{yz} + n\tau_{zz}$$

This is the relation ~~between~~ of the components of stress vector

$\vec{T}^{\nu}$ with the components of stress tensor at any point O (say). Thus we find the stress at any point O is determined completely by the nine components of stress tensor at that point.

Equation of equilibrium of a continuum and proof the symmetry of stress Tensor.

Consider a continuous medium every portion of which contained within the volume B and bounded by a closed surface σ is in equilibrium. Let P be any point whose co-ordinates (x, y, z) within the volume V.



We describe a small closed surface S enclosing P and lying entirely within B. Let V be the volume within the surface S and $\vec{\nu}$, the outward drawn normal to the surface S with d-cs (l, m, n). Let X, Y, Z be the components of body force $\vec{F}$ per unit mass so that $\rho X, \rho Y, \rho Z$ are the components of body force $\rho\vec{F}$ per unit volume. For equilibrium of the matter within volume V the resultant of the body forces within V and the surface forces on S together must vanish. Considering the components of the forces in x-direction we have

$$\iiint_V \rho X\, dv + \iint_S \tau_{\nu x}\, dS = 0 \quad \cdots (1)$$

Now $\tau_{\nu x} = l\tau_{xx} + m\tau_{yx} + n\tau_{zx}$

$$\therefore \iint_S \tau_{\nu x}\, dS = \iint_S (l\tau_{xx} + m\tau_{yx} + n\tau_{zx})\, dS$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right) dv$$

[By Gauss's theorem]

$\therefore$ (1) reduces to

$$\iiint_V \rho X\, dv + \iiint_V \left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right) dv = 0$$

or, $$\iiint_V \left[\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} + \rho X\right] dv = 0$$

Making the dimension of V tends to zero in such a manner that it always encloses the point P, we must have at P

$$\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} + \rho X = 0 \quad \cdots \quad 2.(a)$$

similarly, consider the components of the forces y and z directions respectively, we obtained other two equations

$$\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy} + \rho Y = 0 \quad \cdots \quad 2(b)$$

$$\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} + \rho Z = 0 \quad \cdots \quad 2(c)$$

Problem

Equations 2(a), 2(b), 2(c) are called the <u>equilibrium equation of a continuum.</u>

The material within volume V enclosed by the surface S being in equilibrium the moments of the body and surface forces about x, y and z axes should separately vanish.

Taking moments about the x-axis;

$$\iiint_V \rho(yZ - zY)\,dv + \iint_S (y\tau_{rz} - z\tau_{ry})\,ds = 0 \quad \cdots \quad (3)$$

Now $\displaystyle\iint_S (y\tau_{rz} - z\tau_{ry})\,ds$

$$= \iint_S [y(l\tau_{xz} + m\tau_{yz} + n\tau_{zz}) - z(l\tau_{xy} + m\tau_{yy} + n\tau_{zy})]\,ds$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y\tau_{xz} - z\tau_{xy}) + \frac{\partial}{\partial y}(y\tau_{yz} - z\tau_{yy}) - \tau_{yz} + \frac{\partial}{\partial z}(y\tau_{zz} - z\tau_{zy}) + \tau_{zy}\right] dv$$

$$= \iiint_V \left[y\left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right) - z\left(\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right) - (\tau_{yz} - \tau_{zy})\right] dv$$

$$= \iiint_V [\rho(zY - yZ) + (\tau_{zy} - \tau_{yz})]\,dv \quad [\text{by using } 2(b) \text{ and } 2(c)] \quad \cdots (4)$$

substituting (4) and (3),

$$\iiint_V \rho(yZ - zY)\,dv + \iiint_V [\rho(zY - yZ) + (\tau_{zy} - \tau_{yz})]\,dv = 0.$$

$$\therefore \iiint_V (\tau_{zy} - \tau_{yz})\,dv = 0$$

making the dimension of volume V tends to zero so as to always enclosed the point P, we have $\tau_{zy} = \tau_{yz}$

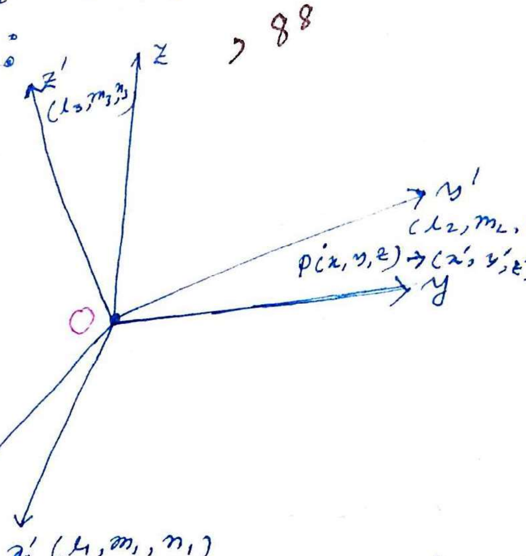
Similarly, taking ... $\tau_{xy} = \tau_{yx}$

...wisely, we set $\tau_{zx} = \tau_{xz}$,

$\therefore$ The stress tensor $\begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$ is symmetric.

Stress Transformation Law :

cos of angle between	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3



Let Ox, Oy, Oz be a set of rectangular axes; Ox', Oy', Oz' are the another set of rectangular axes through O such that d-cs of Ox', Oy', Oz' w.r.t. Ox, Oy, Oz are respectively (l_1, m_1, n_1); (l_2, m_2, n_2) (l_3, m_3, n_3). If P be any point whose co-ordinates referred to Ox, Oy, Oz as axes are (x, y, z) and (x', y', z') be the co-ordinates of the same point referred to Ox', Oy', Oz' as axes then the scheme of transformation is given by table I. We know that if we draw a unit area at O and draw the normal $\vec{r}$ to the surface then the force exerted by the material on the side of the surface towards which normal $\vec{r}$ is drawn to the material on the other side of the surface across the unit area has for its components τ_{rx} τ_{ry}, τ_{rz} where

$$\left.\begin{aligned} \tau_{rx} &= l\tau_{xx} + m\tau_{yx} + n\tau_{zx} \\ \tau_{ry} &= l\tau_{xy} + m\tau_{yy} + n\tau_{zy} \\ \tau_{rz} &= l\tau_{xz} + m\tau_{yz} + n\tau_{zz} \end{aligned}\right\} \cdots (1)$$

l, m, n being the d-cs of the normal $\vec{r}$. If we take $\vec{r}$ in the direction of x' axis,

$$\left.\begin{aligned} \tau_{x'x} &= l_1\tau_{xx} + m_1\tau_{yx} + n_1\tau_{zx} \\ \tau_{x'y} &= l_1\tau_{xy} + m_1\tau_{yy} + n_1\tau_{zy} \\ \tau_{x'z} &= l_1\tau_{xz} + m_1\tau_{yz} + n_1\tau_{zz} \end{aligned}\right\} \cdots (2)$$

$$\therefore \begin{bmatrix} \tau_{x'x} \\ \tau_{x'y} \\ \tau_{x'z} \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \begin{bmatrix} l_1 \\ m_1 \\ n_1 \end{bmatrix} \quad \cdots\cdots (3)$$

Again we take $\vec{r}$ in the direction of y' axis and z' axis we set respectively,

$$\begin{bmatrix} \tau_{y'x} \\ \tau_{y'y} \\ \tau_{y'z} \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \begin{bmatrix} l_2 \\ m_2 \\ n_2 \end{bmatrix} \quad \cdots\cdots (4)$$

$$\begin{bmatrix} \tau_{z'x} \\ \tau_{z'y} \\ \tau_{z'z} \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \begin{bmatrix} l_3 \\ m_3 \\ n_3 \end{bmatrix} \quad \cdots\cdots (5)$$

combining (3), (4), (5), we have

$$\begin{bmatrix} \tau_{x'x} & \tau_{y'x} & \tau_{z'x} \\ \tau_{x'y} & \tau_{y'y} & \tau_{z'y} \\ \tau_{x'z} & \tau_{y'z} & \tau_{z'z} \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \quad \cdots\cdots (6)$$

$\tau_{x'x}$, $\tau_{x'y}$, $\tau_{x'z}$ are the components of force per unit area which the material on the positive side of x'-axis exerts on the material on the negative side across the plane x' = constant.

$\tau_{x'x'}$ is the component of the force per unit area in the direction of x'-axis, which the material on the positive side of x'-axis exerts on the materials on the (-)ve side across the plane x' = constant.

Therefore, $\tau_{x'x'} = l_1 \tau_{x'x} + m_1 \tau_{x'y} + n_1 \tau_{x'z}$

$$= l_1 (l_1 \tau_{xx} + m_1 \tau_{yx} + n_1 \tau_{zx}) + m_1 (l_1 \tau_{xy} + m_1 \tau_{yy} + n_1 \tau_{zy})$$
$$+ n_1 (l_1 \tau_{xz} + m_1 \tau_{yz} + n_1 \tau_{zz}) \quad [\text{from}(3)]$$

$$= l_1^2 \tau_{xx} + m_1^2 \tau_{yy} + n_1^2 \tau_{zz} + 2m_1 n_1 \tau_{yz} + 2n_1 l_1 \tau_{zx} + 2 l_1 m_1 \tau_{xy} \quad \cdots (7)$$

[using $\tau_{xy} = \tau_{yx}$ etc.]

similar formula hold for $\tau_{y'y'}$, $\tau_{z'z'}$.

Now $\tau_{x'y'} = l_2 \tau_{x'x} + m_2 \tau_{x'y} + n_2 \tau_{x'z}$

$$= l_2 (l_1 \tau_{xx} + m_1 \tau_{yx} + n_1 \tau_{zx}) + m_2 (l_1 \tau_{xy} + m_1 \tau_{yy} + n_1 \tau_{zy})$$
$$+ n_2 (l_1 \tau_{xz} + m_1 \tau_{yz} + n_1 \tau_{zz})$$

$$= l_1 l_2 \tau_{xx} + m_1 m_2 \tau_{yy} + n_1 n_2 \tau_{zz} + (m_1 n_2 + m_2 n_1) \tau_{yz} + (n_1 l_2 + n_2 l_1) \tau_{zx}$$
$$+ (l_1 m_2 + l_2 m_1) \tau_{xy} \quad \cdots\cdots (8)$$

Similar formula hold for $\tau_{y'z'}$, $\tau_{z'z'}$.

These are the laws of transformation of stress tensor. We can write this transformation law by the help of matrix. We note that

$$\tau_{x'x'} = l_1 \tau_{x'x} + m_1 \tau_{x'y} + n_1 \tau_{x'z}$$

$$\tau_{x'y'} = l_2 \tau_{x'x} + m_2 \tau_{x'y} + n_2 \tau_{x'z}$$

$$\tau_{x'z'} = l_3 \tau_{x'x} + m_3 \tau_{x'y} + n_3 \tau_{x'z}$$

$$\therefore \begin{bmatrix} \tau_{x'x'} \\ \tau_{x'y'} \\ \tau_{x'z'} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \tau_{x'x} \\ \tau_{x'y} \\ \tau_{x'z} \end{bmatrix}$$

similarly,

$$\begin{bmatrix} \tau_{y'x'} \\ \tau_{y'y'} \\ \tau_{y'z'} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \tau_{y'x} \\ \tau_{y'y} \\ \tau_{y'z} \end{bmatrix}$$

and

$$\begin{bmatrix} \tau_{z'x'} \\ \tau_{z'y'} \\ \tau_{z'z'} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \tau_{z'x} \\ \tau_{z'y} \\ \tau_{z'z} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \tau_{x'x'} & \tau_{y'x'} & \tau_{z'x'} \\ \tau_{x'y'} & \tau_{y'y'} & \tau_{z'y'} \\ \tau_{x'z'} & \tau_{y'z'} & \tau_{z'z'} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \tau_{x'x} & \tau_{y'x} & \tau_{z'x} \\ \tau_{x'y} & \tau_{y'y} & \tau_{z'y} \\ \tau_{x'z} & \tau_{y'z} & \tau_{z'z} \end{bmatrix} \quad \cdots (9)$$

Taking the help of (6), (9) becomes

$$\begin{bmatrix} \tau_{x'x'} & \tau_{y'x'} & \tau_{z'x'} \\ \tau_{x'y'} & \tau_{y'y'} & \tau_{z'y'} \\ \tau_{x'z'} & \tau_{y'z'} & \tau_{z'z'} \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \quad \cdots (10)$$

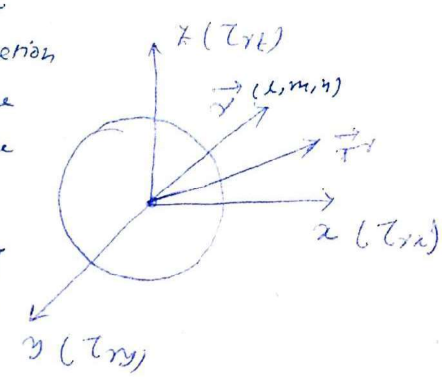
If the subscripts x, y, z of the components of stress tensor be replaced by 1, 2, 3 respectively and x', y', z' be replaced by $1', 2', 3'$, then equation (10) can be written as

$$[\tau_{i'j'}] = [\alpha][\tau_{ij}][\alpha]^T, \quad i, j = 1, 2, 3 \quad \cdots (11) - 8)$$

where $[\tau_{i'j'}]$ is the matrix consisting of the nine components of stress in the ~~principal axis~~ primed axis, $[\tau_{ij}]$ is the matrix of stress component corresponding to unprimed axis and $[\alpha]$ is the matrix formed by the d.c.s of ox', oy', oz' as elements and $[\lambda]^T$ is its transpose. Equation (10) or (11) gives the laws of transformation of stress tensor in matrix form.

Principal stresses and principal axes of stress: 994

Let us consider a unit area in the elastic body and let $\vec{\gamma}$ be the unit vector in the direction of normal to it in a specified sense. The force exerted by the material towards the side of $\vec{\gamma}$ on the material on the opposite side across the unit area has for its components $\tau_{\gamma x}, \tau_{\gamma y}, \tau_{\gamma z}$. The resultant of these three forces has a magnitude and direction and we denote it by means of the vector $\vec{T}^{\gamma}$ as shown the figure.



In general, the direction of $\vec{T}^{\gamma}$ is different from the direction $\vec{\gamma}$. But if the orientation of the unit area be such that $\vec{T}^{\gamma}$ is in the direction $\vec{\gamma}$ then this direction is called principal direction of stress. For a principal stress direction $\vec{T}^{\gamma} = \sigma\vec{\gamma}$ in which σ, the magnitude of stress vector is called principal stress vector.

If $\vec{\gamma}$ be the direction of principal stress and σ be the value of principal stress then according to definition of

$\tau_{\gamma x} = l\sigma, \quad \tau_{\gamma y} = m\sigma, \quad \tau_{\gamma z} = n\sigma$ where l, m, n are the d-cs of $\vec{\gamma}$.

$$\therefore \left.\begin{aligned} l\tau_{xx} + m\tau_{yx} + n\tau_{zx} &= l\sigma \\ l\tau_{xy} + m\tau_{yy} + n\tau_{zy} &= m\sigma \\ l\tau_{xz} + m\tau_{yz} + n\tau_{zz} &= n\sigma \end{aligned}\right\} \quad \text{---- (1)}$$

$$\text{or, } \left.\begin{aligned} l(\tau_{xx}-\sigma) + m\tau_{yx} + n\tau_{zx} &= 0 \\ l\tau_{xy} + m(\tau_{yy}-\sigma) + n\tau_{zy} &= 0 \\ l\tau_{xz} + m\tau_{yz} + n(\tau_{zz}-\sigma) &= 0 \end{aligned}\right\} \quad \text{---- (2)}$$

The system of eq^n (2) has a set of non-vanishing solutions l, m, n iff. the determinant of the coefficient vanish,

$$\text{i.e.} \begin{vmatrix} \tau_{xx}-\sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy}-\sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz}-\sigma \end{vmatrix} = 0 \quad \text{----- (3)}$$

It is a cubic eq^n in σ having three values of σ. Let these roots be $\sigma_1, \sigma_2, \sigma_3$ they are called the three principal stress values. Associated with each of the principal stresses $\sigma_1, \sigma_2, \sigma_3$ there are three principal stress direction having d-cs