$$\frac{DE}{Dt} = \frac{d}{dt}\iiint_V \rho e\,dv = \iiint_V \left[\frac{D}{Dt}(\rho e) + \rho e\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)\right] dv$$

$$= \iiint_V \left[e\left\{\frac{D\rho}{Dt} + \rho\frac{\partial v_i}{\partial x_i}\right\} + \rho\frac{De}{Dt}\right] dv$$

$$= \iiint_V \rho\frac{De}{Dt}\,dv \quad \ldots (3)$$

[since quantity in the 2nd bracket is zero by the eq^n of continuity]

Rate of work done by body and surface forces on the material in V $= \iiint_V \rho(X_1 v_1 + X_2 v_2 + X_3 v_3)\,dv + \iint_S (\tau_{r1} v_1 + \tau_{r2} v_2 + \tau_{r3} v_3)\,ds$

$$= \iiint_V \rho X_j v_j\,dv + \iint_S [(l\tau_{11} v_1 + m\tau_{21} v_1 + n\tau_{31} v_1) + (l\tau_{12} v_2 + m\tau_{22} v_2 + n\tau_{32} v_2) + (l\tau_{13} v_3 + m\tau_{23} v_3 + n\tau_{33} v_3)]\,ds$$

$$= \iiint_V X_j v_j \rho\,dv + \iint_S [(l\tau_{1j} v_j + m\tau_{2j} v_j + n\tau_{3j} v_j)]\,ds$$

$$= \iiint_V X_j v_j \rho\,dv + \iiint_V \frac{\partial}{\partial x_i}(\tau_{ij} v_j)\,dv, \text{ [by Gaussian theorem]}$$

$\therefore$ Rate of work done by body and surface forces on the material in V $= \iiint_V X_j v_j \rho\,dv + \iiint_V \left[v_j\frac{\partial \tau_{ij}}{\partial x_i} + \tau_{ij}\frac{\partial v_j}{\partial x_i}\right] dv \ldots (4)$

If 'h' be the body heat energy or radiant heat energy generated per unit mass per unit time and vector $\vec{c}$, $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ represent the flow of heat per unit area across a surface per unit time, the rate of ~~increase~~ of total heat energy into the continuum enclosed in V $= \iiint_V \rho h\,dv - \iint_S (lc_1 + mc_2 + nc_3)\,ds$

$$= \iiint_V \rho h\,dv - \iiint_V \frac{\partial c_i}{\partial x_i}\,dv \quad \ldots (5)$$

Substituting ~~these~~ results (2), (3), (4) and (5) in (1), we get

$$\iiint_V \rho v_j \frac{Dv_j}{Dt}\,dv + \iiint_V \rho\frac{De}{Dt} = \iiint_V \rho v_j X_j\,dv + \iiint_V \left[v_j\frac{\partial \tau_{ij}}{\partial x_i}\right.$$

$$\iiint_V \rho v_j \frac{Dv_j}{Dt}\,dv + \iiint_V \rho \frac{De}{Dt}\,dv = \iiint \rho v_j X_j\,dv$$

$$+ \iiint_V \left[v_j \frac{\partial \tau_{ij}}{\partial x_i} + \tau_{ij}\frac{\partial v_j}{\partial x_i}\right] dv + \iiint_V \rho h\,dv - \iiint_V \frac{\partial c_i}{\partial x_i}\,dv$$

or $$\iiint_V v_j\left[\rho \frac{Dv_j}{Dt} - \rho X_j - \frac{\partial}{\partial x_i}\tau_{ij}\right] dv + \iiint_V \left[\rho\frac{De}{Dt} - \rho h + \frac{\partial c_i}{\partial x_i} - \tau_{ij}\frac{\partial v_j}{\partial x_i}\right] dv = 0 \quad \cdots (6)$$

since the eq^n of motion is given by

$$\rho \frac{Dv_j}{Dt} = \rho X_j + \frac{\partial \tau_{ij}}{\partial x_i}. \quad \cdots (7)$$

so, first integral on the R.H.S. of (7) is zero.

therefore, $$\iiint_V \left[\rho\frac{De}{Dt} - \rho h + \frac{\partial c_i}{\partial x_i} - \tau_{ij}\frac{\partial v_j}{\partial x_i}\right] dv = 0 \quad \cdots (8)$$

since the integral is zero for any ~~value~~ volume V so that the integrand ~~is zero~~ must vanish i.e.

$$\rho \frac{De}{Dt} = \rho h - \frac{\partial c_i}{\partial x_i} + \tau_{ij}\frac{\partial v_j}{\partial x_i} \quad \cdots (9)$$

Now $$\tau_{ij}\frac{\partial v_j}{\partial x_i} = \frac{1}{2}\tau_{ij}\left[\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right] + \frac{1}{2}\tau_{ij}\left[\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right] \cdots (10)$$

$$= \cancel{\tfrac{1}{2}}\,\tau_{ij} D_{ij} + 0 \quad \cdots (11)$$

where $D_{ij} = \frac{1}{2}\left[\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right]$ is the strain-stress tensor.

The last term on the R.H.S. of (10) vanishes because it is the product of a symmetric Tensor with an anti-symmetric one. Using the result (11) in (9), we obtain the final form of the energy equation

$$\rho\frac{De}{Dt} = \rho h - \frac{\partial c_i}{\partial x_i} + \tau_{ij} D_{ij} \quad \cdots (12)$$

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Note: In case of absence of thermal energy the eq^n becomes

$$\rho \frac{De}{Dt} = \tau_{ij} D_{ij}$$

$$D_{ij} = \frac{1}{2}\left[\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x_i}\cdot\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\right] = \frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right] = \frac{\partial}{\partial t} e_{ij} = \dot{e}_{ij}$$

where u_i and u_j are compts. of displacement

Strain energy function

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$V_0 \to$ vol. at time $t=0$ occupying a set of material particles

$V \to$ new vol. at time t occupying the same set of material particles

$S \to$ bounding surface of V.

$u_1, u_2, u_3 \to$ displacement components.

$v_1, v_2, v_3 \to$ velocity components at any pt. (x_1, x_2, x_3).

$\rho \to$ density at (x_1, x_2, x_3)

$\rho X_1, \rho X_2, \rho X_3 \to$ components of body force per unit vol.

$l, m, n \to$ d-cs of outward drawn normal to the surface S at any point.

Let K be the K.E. of the material within V, then

$$K = \frac{1}{2} \iiint_V \rho (v_1^2 + v_2^2 + v_3^2)\, dV$$

or,
$$\frac{dK}{dt} = \frac{d}{dt} \iiint_V \frac{\rho}{2} (v_1^2 + v_2^2 + v_3^2)\, dV$$

$$= \iiint_V \Big[\frac{1}{2} \frac{D}{Dt}(\rho v_1^2) + \frac{1}{2}\frac{D}{Dt}(\rho v_2^2) + \frac{1}{2}\frac{D}{Dt}(\rho v_3^2) + \left(\frac{\rho v_1^2}{2} + \frac{\rho v_2^2}{2} + \frac{\rho v_3^2}{2}\right)\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)\Big] dV$$

$$= \iiint_V \Big[\rho \Big\{ v_1 \frac{Dv_1}{Dt} + v_2 \frac{Dv_2}{Dt} + v_3 \frac{Dv_3}{Dt} \Big\} + \frac{q^2}{2} \Big\{ \frac{D\rho}{Dt} + \rho \operatorname{div} \vec{q} \Big\} \Big] dV$$

where $\vec{q} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

the quantity $\frac{D\rho}{Dt} + \rho \operatorname{div} \vec{q} = 0$ by the eq^n of continuity

so,
$$\frac{dK}{dt} = \iiint_V \rho v_j \frac{Dv_j}{Dt}\, dV$$

From the eq^n of motion of continuum we have

$$\rho \frac{Dv_j}{Dt} = \rho X_j + \frac{\partial \tau_{ij}}{\partial x_i}$$

so,
$$\frac{dK}{dt} = \iiint_V \rho X_j v_j \, dV + \iiint_V v_j \frac{\partial \tau_{ij}}{\partial x_i}\, dV$$

$$= \iiint_V \rho X_j v_j\, dV + \iiint_V \frac{\partial}{\partial x_i}(v_j \tau_{ij})\, dV - \iiint_V \tau_{ij} \frac{\partial v_j}{\partial x_i} dV \quad \ldots (1)$$

1st integral on R.H.S. ... the rate of work done by body forces. The 2nd integral $= \iiint_V \frac{\partial}{\partial x_i}(v_j \tau_{ij})\, dV$

$$= \iiint_V \Big[\frac{\partial}{\partial x_1}(v_1 \tau_{11} + v_2 \tau_{12} + v_3 \tau_{13}) + \frac{\partial}{\partial x_2}(v_1 \tau_{21} + v_2 \tau_{22} + v_3 \tau_{23}) + \frac{\partial}{\partial x_3}(v_1 \tau_{31} + v_2 \tau_{32} + v_3 \tau_{33}) \Big] dV$$

$$= \iint_S \Big[l(v_1 \tau_{11} + v_2 \tau_{12} + v_3 \tau_{13}) + m(v_1 \tau_{21} + v_2 \tau_{22} + v_3 \tau_{23}) + n(v_1 \tau_{31} + v_2 \tau_{32} + v_3 \tau_{33}) \Big] dS$$

[by Gauss's theorem]

$$= \iint_S \Big[v_1(l\tau_{11} + m\tau_{21} + n\tau_{31}) + v_2(l\tau_{12} + m\tau_{22} + n\tau_{32}) + v_3(l\tau_{13} + m\tau_{23} + n\tau_{33}) \Big] dS$$

$$= \iint_S (v_1 \tau_{\nu 1} + v_2 \tau_{\nu 2} + v_3 \tau_{\nu 3})\, dS$$

$$= \iint_S v_j \tau_{\nu j}\, dS = \text{rate of work done by surface forces.}$$

For the 3rd integral on the R.H.S. of ①

$$\tau_{ij} \frac{\partial v_j}{\partial x_i} = \frac{1}{2}\tau_{ij}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) + \frac{1}{2}\tau_{ij}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)$$

$$= \tau_{ij} D_{ij} + 0$$

where D_{ij} is the strain rate tensor i.e. $\dot{e}_{ij}$ where e_{ij} a strain tensor and dot $(\cdot)$ indicates differentiation w.r.t. t.

So, the 3rd integral $-\iiint_V \tau_{ij} \frac{\partial v_j}{\partial x_i}\, dV = -\iiint_V \tau_{ij} \dot{e}_{ij}\, dV$

Using these results in eqn ① we obtain

$$\frac{dK}{dt} + \iiint_V \tau_{ij} \dot{e}_{ij}\, dV = \iiint_V \rho X_j v_j\, dV + \cancel{\iiint_V \frac{\partial}{\partial x_i}(v_j \tau_{ij})\, dV} + \iint_S v_j \tau_{\nu j}\, dS$$

= rate of work done by body force + rate of work done by surface force

But in the absence of any thermal effect the principle of conservation of energy gives rate of increase of K.E. in V + rate of increase of internal energy within V

= rate of work done by body force + rate of work done by surface force

$$\therefore \frac{dK}{dt} + \iiint_V \rho \frac{de}{dt}\, dV = \iiint_V \rho X_j v_j \, dV + \iint_S v_j \tau_{kj}\, dS$$

where $\frac{de}{dt}$ is the rate of increase of internal energy per unit mass.

since, $\frac{dK}{dt} + \iiint_V \tau_{ij} \dot{e}_{ij}\, dV = \frac{dK}{dt} + \iiint_V \rho \frac{de}{dt}\, dV$

$$\Rightarrow \boxed{\frac{de}{dt} = \frac{\tau_{ij}}{\rho} \cdot \dot{e}_{ij}}$$

e, the internal energy per unit mass is purely mechanical in this case, and is called the strain energy per unit mass.

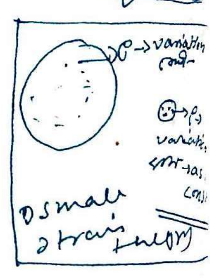
or, $de = \frac{\tau_{ij}}{\rho} \cdot de_{ij}$

If e is considered as a function of strain components then $e = e(e_{ij})$.

Its differential is given by de

$$de = \frac{\partial e}{\partial e_{11}} de_{11} + \frac{\partial e}{\partial e_{12}} de_{12} + \cdots + \frac{\partial e}{\partial e_{23}} de_{23} + \cdots + \frac{\partial e}{\partial e_{33}} de_{33}$$

$$= \frac{\partial e}{\partial e_{ij}} de_{ij}$$



so, $\frac{\tau_{ij}}{\rho} = \frac{\partial e}{\partial e_{ij}}$

Let U be the strain energy per unit vol., then

$U = \rho e$.

In small strain theory ρ may be considered constant.

Therefore, $\tau_{ij} = \rho \frac{\partial e}{\partial e_{ij}} = \frac{\partial(\rho e)}{\partial e_{ij}} = \frac{\partial U}{\partial e_{ij}}$ ✓✓

U is called the strain energy density function.

Let us assume that strain energy density function $U(e_{ij})$ & can be expressed in a power series in terms of e_{ij}'s.

[power series → oral
Analytic fn → oral]

$U(e_{ij}) = C_0 + C_{mn} e_{mn} + \frac{1}{2} C_{pqrs} e_{pq} e_{rs} + \cdots\cdots$

discarding all terms of order 3 and higher in the strains in the expansion of U.

since, $\tau_{ij} = \frac{\partial U}{\partial e_{ij}}$

so, $\tau_{ij} = \frac{\partial U}{\partial e_{ij}} = C_{ij} + \frac{1}{2}\left(C_{ijrs} e_{rs} + C_{pqij} e_{pq}\right)$

$= C_{ij} + \frac{1}{2}\left(C_{ijrs} + C_{rsij}\right) e_{rs}$

For the stress to vanish in the absence of strains the const. C_{ij} must be equal to zero. Thus the expression for strain energy density function reduces to $U = \frac{1}{2} C_{pqrs} e_{pq} e_{rs}$

??2

or, $U = \frac{1}{2} C_{ijrs} e_{ij} e_{rs}$

Then the expression for stress is $\tau_{ij} = \frac{1}{2}\left(C_{ijrs} + C_{rsij}\right) e_{rs}$

The tensor $\frac{1}{2}\left(C_{ijrs} + C_{rsij}\right)$ is symmetric w.r.t. (ij) and (rs) i.e. if (rs) is replaced by (ij) and (ij) be replaced by (rs) then it remains unchanged.

we put $A_{ijrs} = \frac{1}{2}\left(C_{ijrs} + C_{rsij}\right)$

$\therefore \tau_{ij} = A_{ijrs} e_{rs}$

So the co-efficients in the generalised Hooke's law are symmetric if strain energy density function U exists.

So the existence of strain energy function reduces the no. of co-efficients from 36 to 21 in generalised Hooke's law.

we write strain energy function $U = \frac{1}{2} C_{ijrs} e_{ij} e_{rs}$ in the form $U = \frac{1}{2}\left[\frac{1}{2}\left(C_{ijrs} + C_{rsij}\right) + \frac{1}{2}\left(C_{ijrs} - C_{rsij}\right)\right] e_{ij} e_{rs}$

let $\frac{1}{2}\left(C_{ijrs} - C_{rsij}\right) = B_{ijrs}$. we note that tensor B_{ijrs} is anti-symmetric w.r.t. (ij) and (rs).

i.e. $B_{ijrs} = -B_{rsij}$

[Therefore $U = \frac{1}{2}\left[A_{ijrs} - B_{rsij}\right] e_{ij} e_{rs}$]

Therefore, $U = \frac{1}{2} A_{ijrs} e_{ij} e_{rs} + 0$

$B_{ijrs} e_{ij} e_{rs} = 0$ because it is the product of a symmetric tensor $(e_{ij} e_{rs})$ with an antisymmetric tensor B_{ijrs}.

$\therefore \quad U = \frac{1}{2} A_{ijrs}\, e_{ij}\, e_{rs}$

So, using ~~Compairing~~ with $\tau_{ij} = A_{ijrs}\, e_{rs}$, $\quad U = \frac{1}{2}\tau_{ij}\, e_{ij}$ '90 '89

This formula for strain energy density function is known as the <u>Clapeyron formula</u>. '92 '89

When the stress-strain relation law is written in the form $e_{ij} = S_{ijmn}\,\tau_{mn}$

where $S_{ijmn} = S_{mnij}$, then the Clapeyron formula gives

$$U = \frac{1}{2} S_{ijmn}\,\tau_{ij}\,\tau_{mn} = \cancel{\frac{1}{2} S_{pqmn}\,\tau_{pq}\,\tau_{mn}}$$

$$\cancel{\therefore \frac{\partial U}{\partial \tau_{ij}} = \frac{1}{2} S_{ijmn}\,\tau_{mn} + \frac{1}{2} S_{psij}\,\tau_{pq}} \; \cancel{S_{ijmn}\,\tau_{mn}} = e_{ij}$$

~~This is Castigliano's formula.~~

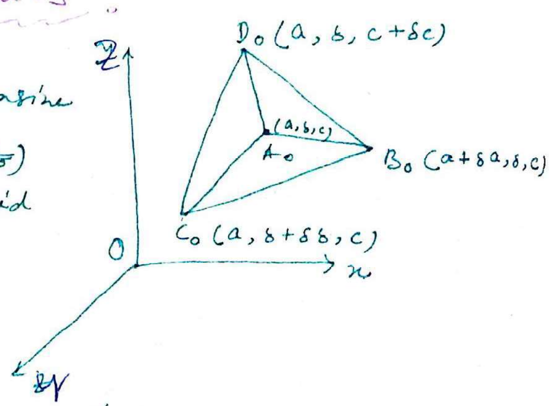
$$\frac{\partial U}{\partial \tau_{ij}} = \frac{1}{2} S_{ijmn}\,\tau_{mn} + \frac{1}{2} S_{pqij}\,\tau_{pq}$$

$$\therefore \frac{\partial U}{\partial \tau_{ij}} = S_{ijmn}\,\tau_{mn} = e_{ij}$$

This is Castigliano's formula. '92 '89

*** Equation of continuity using Lagrange's system of co-ordinates:

'89 '92



With reference to this system imagine a small tetrahedron ~~(A_0, B_0, C_0, D_0)~~ $A_0B_0C_0D_0$ lying within the fluid with its edges A_0B_0, A_0C_0, A_0D_0 perpendicular to the co-ordinate axes of lengths δa, δb, δc respectively. Let $A_0B_0C_0D_0$ after the lapse of time t form a differently situated tetrahedron ABCD.

Let the co-ordinates of A_0 initially be (a, b, c) occupy the position A with co-ordinates (x, y, z) after a time 't'. Obviously x, y, z are the functions of t and initial values a, b, c so that

$$x = f(a, b, c, t)$$
$$y = g(a, b, c, t)$$
$$z = h(a, b, c, t)$$

so x co-ordinates of B is $f(a+\delta a, b, c, t) = f(a,b,c,t) + \frac{\partial f}{\partial a}\delta a$

$$= x + \frac{\partial x}{\partial a}\delta a$$

and the y-coordinates of B is [neglecting terms involving higher power of δa]

$$g(a+\delta a, b, c, t) = g(a,b,c,t) + \frac{\partial g}{\partial a}.\delta a$$

$$= y + \frac{\partial y}{\partial a}.\delta a$$

and the z-coordinates of B

is $h(a+\delta a, b, c, t) = h(a,b,c,t) + \frac{\partial h}{\partial a}\delta a$

$$= z + \frac{\partial z}{\partial a}\delta a$$

so co-ordinates of B relative to A are $\frac{\partial x}{\partial a}\delta a, \frac{\partial y}{\partial a}\delta a, \frac{\partial z}{\partial a}\delta a$.

similarly, the co-ordinates of C relative to A are

$$\frac{\partial x}{\partial b}\delta b, \frac{\partial y}{\partial b}\delta b, \frac{\partial z}{\partial b}\delta b.$$

and the co-ordinates of D relative to A are $\frac{\partial x}{\partial c}\delta c, \frac{\partial y}{\partial c}\delta c, \frac{\partial z}{\partial c}\delta c$.

Now the vol. of the tetrahedron $A_0B_0C_0D_0 = \frac{1}{6}\delta a\,\delta b\,\delta c$.

Volume of the tetrahedron $ABCD$

$$= \frac{1}{6}\begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix}\delta a\,\delta b\,\delta c$$

$$= \frac{1}{6}\cdot\frac{\partial(x,y,z)}{\partial(a,b,c)}.\delta a\,\delta b\,\delta c$$

$$= \frac{1}{6}J\,\delta a\,\delta b\,\delta c$$, where J is the Jacobian.

If the tetrahedron be taken, so small that the density of the liquid within the tetrahedron may be taken to be constant, and if density of liquid at A_0 be ρ_0 and if density of liquid at A be ρ then mass of liquid within the tetrahedron. $A_0B_0C_0D_0 = \frac{1}{6}\rho_0\,\delta a\,\delta b\,\delta c$.

$\therefore$ mass of the ~~tetrahedron~~ liquid within the tetrahedron

$ABCD = \frac{1}{6} \rho . J \delta a \delta b \delta c$.

From the principle of conservation of mass

$$\frac{1}{6} \rho_0 \delta a \delta b \delta c = \frac{1}{6} \rho J \delta a \delta b \delta c$$

$$\Rightarrow \rho_0 = \rho J = \text{Const.}$$

This is the equation of continuity in Lagrangian form.

Equivalence of equations of continuity in Lagrangian and Eulerian co-ordinates: 289

In Lagrangian form equation of continuity is

$$\rho J = \text{Const.} = \rho_0 \quad \text{where } J = \frac{\partial(x, y, z)}{\partial(a, b, c)}$$

$$\therefore \frac{d}{dt}(\rho J) = 0$$

$$\Rightarrow \cancel{J} \; J \frac{d\rho}{dt} + \rho \frac{dJ}{dt} = 0 \; \cdots\cdots (1)$$

So to pass from Lagrangian to Eulerian we replace $\frac{d}{dt}$ by $\frac{D}{Dt}$ and $\dot{x}, \dot{y}, \dot{z}$ by u, v, w respectively.

Now
$$\frac{dJ}{dt} = \frac{d}{dt} \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial a}\frac{dx}{dt} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial}{\partial b}\cdot\frac{dx}{dt} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial}{\partial c}\cdot\frac{dx}{dt} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial}{\partial a}\cdot\frac{dy}{dt} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial}{\partial b}\cdot\frac{dy}{dt} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial}{\partial c}\cdot\frac{dy}{dt} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial}{\partial a}\frac{dz}{dt} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial}{\partial b}\cdot\frac{dz}{dt} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial}{\partial c}\cdot\frac{dz}{dt} \end{vmatrix}$$

putting $u = \frac{dx}{dt} = \dot{x}$ in the 1st determinant, $\cdots (2)$

$$\frac{\partial(u, y, z)}{\partial(a, b, c)} = \frac{\partial(u, y, z)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(a, b, c)}$$

$$= J \begin{vmatrix} \frac{\partial u}{\partial x} & 0 & 0 \\ \frac{\partial u}{\partial y} & 1 & 0 \\ \frac{\partial u}{\partial z} & 0 & 1 \end{vmatrix} = J \frac{\partial u}{\partial x}$$

Similarly from 2nd and 3rd determinant of (2), we get $J \frac{\partial v}{\partial y}$, $J \frac{\partial w}{\partial z}$.

So eqn ① becomes,

$$J \frac{D\rho}{Dt} + \rho J \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

or, $$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

or, $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0.$$

which is the equation of continuity in Eulerian form.

now Again class note follows	After Castigliano's Formula	CHAP V	up to date
B.D.	1st July		B.D.

FLUID DYNAMICS*

General analysis of fluid motion:

Let P be any pt. of the fluid with co-ordinates (x, y, z). Let Q be a neighbouring pt. of P with co-ordinates $(x+X, y+Y, z+Z)$. X, Y, Z are taken so small that their squares and products may be neglected. Let u, v, w be the components of velocity of fluid particle at P at time t. Then u, v, w are functions of x, y, z and t.

$\therefore \quad u = f(x, y, z, t), \quad v = g(x, y, z, t), \quad w = h(x, y, z, t)$

If $u + \delta u$, $v + \delta v$, $w + \delta w$ be the components of velocity of the fluid at Q at time t then

$$u + \delta u = f(x+X,\ y+Y,\ z+Z,\ t)$$

$$= f(x,y,z,t) + \frac{\partial f}{\partial x}X + \frac{\partial f}{\partial y}Y + \frac{\partial f}{\partial z}Z + \cdots$$

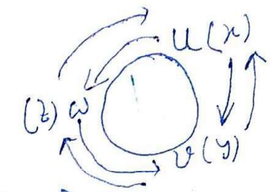
$$= u + \frac{\partial u}{\partial x}X + \frac{\partial u}{\partial y}Y + \frac{\partial u}{\partial z}Z$$

$$\Rightarrow \quad \delta u = \frac{\partial u}{\partial x}X + \frac{\partial u}{\partial y}Y + \frac{\partial u}{\partial z}Z$$

similarly, $\delta v = \frac{\partial v}{\partial x}X + \frac{\partial v}{\partial y}Y + \frac{\partial v}{\partial z}Z$

AND $\delta w = \frac{\partial w}{\partial x}X + \frac{\partial w}{\partial y}Y + \frac{\partial w}{\partial z}Z$

Neglecting higher derivative terms



writing these terms in the form

$$\delta u = \frac{\partial u}{\partial x}X + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)Y + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)Z$$
$$- \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)Y + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)Z$$

$$\delta v = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)X + \cancel{\tfrac{1}{2}}\frac{\partial v}{\partial y}Y + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)Z$$
$$+ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)X - \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)Z$$

$$\delta w = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)X + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)Y + \frac{\partial w}{\partial z}Z$$
$$- \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)X + \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)Y$$

Let us put $\frac{\partial u}{\partial x} = a$, $\frac{\partial v}{\partial y} = b$, $\frac{\partial w}{\partial z} = c$

$$F = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad G = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$H = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \quad \xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

then δu, δv, δw consists of two parts: 1stly

$$(\delta u)_1 = 0\cdot X - \zeta Y + \eta Z$$

$$(\delta v)_1 = \zeta X + 0\cdot Y - \xi Z$$

$(\delta w)_1 = -\eta x + \xi y + 0.z$

these are the components of the velocity of the fluid element at Q rotating as if rigid body about the pt. P with angular velocity having components ξ, η, ζ about the co-ordinate axes.

The 2nd part of $\delta u, \delta v, \delta w$ are

$(\delta u)_2 = a.x + H.y + G.z$

$(\delta v)_2 = H.x + b y + F z$

$(\delta w)_2 = G x + F y + c z$

this part of the velocity is normal to the surface

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2h\cancel{xy} = \text{Const.}$

passing through ~~P~~ Q.

therefore the motion of the small mass of fluid surrounding the pt. P consists of three parts

(i) the motion of translation u, v, w.

(ii) rigid body rotation about P with angular velocity having components ξ, η, ζ.

(iii) A motion normal to the surface $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = \text{const.}$ and passing through Q.

the 1st two are rigid body motion and the (iii) represents the deformation of mass of fluid

Stream Line in a fluid*

A stream line or a line of flow at any instant is a line drawn in the fluid such that the tangent to this line at any pt. is the direction of the velocity of the ~~fluid~~ fluid at that point.

Let the direction of motion of a fluid particle at the point (x, y, z) is given by the components of velocity u, v, w s.t. $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and

$\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$, the differential eqⁿ of the stream lines are given by

$$\vec{q} \times d\vec{r} = \vec{0}$$

i.e. $(u\vec{i} + v\vec{j} + w\vec{k}) \times (dx\vec{i} + dy\vec{j} + dz\vec{k}) = \vec{0}$

or, $(v\,dz - w\,dy)\vec{i} + (w\,dx - u\,dz)\vec{j} + (u\,dy - v\,dx)\vec{k} = \vec{0}$

so that $v\,dz - w\,dy = 0$, $w\,dx - u\,dz = 0$, $u\,dy - v\,dx = 0$

$\therefore \quad \dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w} \quad \ldots (1)$ is the differential eqⁿ of the stream line

$\boxed{\text{Ex}}$ Obtain the stream line of a flow $u = x, v = -y$.

Solⁿ In this case, $\vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$

In general $\qquad = x\vec{i} - y\vec{j}$

Eqⁿˢ of stream lines are $\dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w}$

i.e. $\dfrac{dx}{x} = \dfrac{dy}{-y}$

$\Rightarrow$ ~~logx = logy~~ $\dfrac{dx}{x} + \dfrac{dy}{y} = 0$ and $dz = 0$

Int. $xy = c_1$ and $z = c_2$

$\therefore$ the reqd. stream lines are given by the curves of intersection of $xy = c_1$ and $z = c_2$.