It can be shown that for an isotropic material the above mentioned the 36 elastic constants are reduced to only two independent constants namely viz. λ and μ which are known as <u>Lame's constants</u> for an isotropic material we have the following stress-strain relations ~~relations~~

$$\tau_{xx} = \lambda\theta + 2\mu \frac{\partial u_x}{\partial x}$$

$$\tau_{yy} = \lambda\theta + 2\mu \frac{\partial u_y}{\partial y}$$

$$\tau_{zz} = \lambda\theta + 2\mu \frac{\partial u_z}{\partial z}$$

$$\tau_{yz} = \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)$$

$$\tau_{zx} = \mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)$$

$$\tau_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

where $\theta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$

$= e_{xx} + e_{yy} + e_{zz}$

Constitutive equation for viscous fluid: (Firctional Force)

Relation between stresses and rate of strain in viscous fluid: If u, v, ω be the components of velocity at any point of viscous fluid then for an isotropic fluid we have the experimental relations

$$\tau_{xx} = -p + \lambda\dot{\theta} + 2\mu \frac{\partial u}{\partial x}$$

$$\tau_{yy} = -p + \lambda\dot{\theta} + 2\mu \frac{\partial v}{\partial y}$$

$$\tau_{zz} = -p + \lambda\dot{\theta} + 2\mu \frac{\partial \omega}{\partial z}$$

where $\dot{\theta} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z}$, p is the mean pressure at any point and λ, μ are physical constants associated with fluid.

Also we have the relations

$$\tau_{yz} = \mu\left(\frac{\partial \omega}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \mu\left(\frac{\partial \omega}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

(z) ω $u(x)$ $v(y)$

From the 1st three relations we have by addition

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = -3p + (3\lambda + 2\mu)\dot{\theta}$$

From the def^n of mean pressure we have by addition

$\tau_{xx} + \tau_{yy} + \tau_{zz} = -3p$

$\therefore$ we have, $(3\lambda + 2\mu)\dot{\theta} = 0$

In case of a incompressible fluid (liquid) $\dot{\theta} = 0$

and λ disappears from the stress-rate of strain relations.

In case of a compressible fluid (gas)

$3\lambda + 2\mu = 0$ i.e. $\lambda = -\frac{2}{3}\mu$

This is stoke's relation for gases. μ is called the coefficient of viscosity of fluid.

[Taylor's series for a function of several variables]

$f_1(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots, x_n + \Delta x_n)$

$$= f(x_1, x_2, \cdots, x_n) + \frac{\partial f}{\partial x_1} \cdot \Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n$$

$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x_1^2} \cdot (\Delta x_1)^2 + \cdots + \frac{\partial^2 f}{\partial x_n^2}(\Delta x_n)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + \cdots \right.$$

$$\left. \cdots + 2\frac{\partial^2 f}{\partial x_{n-1} x_n}\Delta x_{n-1}\Delta x_n\right] + \cdots$$

* Linear visco-elastic behaviour:

Elastic solids and viscous fluids differ widely in their deformational characteristics. Elastically deformed bodies return to a natural or underformed state upon removal of applied loads. viscous fluids possess no tendency at all for deformational recovery. Also elastic stress is related directly to deformation where as stress in a viscous fluid depends upon rate of deformation.

Gauss's theorem:

Let u, v, w be three single-valued functions of (x, y, z) which together with 1st derivatives are continuous on a closed region R and on its boundary S. Let l, m, n be the d-cs of the out ward drawn normal to the surface S at any point then

$$\iiint_R \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] dx\,dy\,dz = \iint_S (lu + mv + nw)\,dS$$

"MOTION OF DEFORMABLE BODIES"

In order to study the motion of a continuum we usually follow one of the two methods

(i) Lagrangian description of motion $\boxed{or}$ Material description of motion of a continuum.

(ii) Eulerian description of motion $\boxed{or}$ Spatial description of a motion of a continuum.

(i) Let a fixed frame of reference $Oxyz$ be chosen. Let a material particle which is initially at P_0 with co-ordinates (a,b,c) move to another point (x,y,z) after the lapse of time t. The co-ordinate (x,y,z) will be a function of t and there initial values (a,b,c). ~~Therefore~~

$$\therefore \quad x = f(a,b,c,t) \; ; \; y = g(a,b,c,t) \; ; \; z = h(a,b,c,t)$$

Components of velocity of the particle at time t whose initial co-ordinates is (a,b,c), is $\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}$ and the accelerations are $\frac{\partial^2 x}{\partial t^2}, \frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 z}{\partial t^2}$. In this differentiation w.r.t. t the naming co-ordinates (a,b,c) are kept unaltered. Such differentiation is refered to as "particle differentiation or differentiation following the particle".

Lagrangian description or material description of motion is usually used in elastic solids. Let (X,Y,Z) be the co-ordinates of a particle of an elastic solid at time $t=0$. If the displacement of this particle at time t be u_x, u_y, u_z then its co-ordinates after time t are

$$x = X + u_x \; ; \; y = Y + u_y \; ; \; z = Z + u_z .$$

The displacement components u_x, u_y, u_z are functions of initial position (X,Y,Z) and t. Therefore position of this particle (x,y,z) is also function of initial position (X,Y,Z) and t. Therefore its velocity components $\dot{x}, \dot{y}, \dot{z}$ are $\frac{\partial u_x}{\partial t}, \frac{\partial u_y}{\partial t}, \frac{\partial u_z}{\partial t}$.

And the acceleration components are $\frac{\partial^2 u_x}{\partial t^2}, \frac{\partial^2 u_y}{\partial t^2}, \frac{\partial^2 u_z}{\partial t^2}$.

During differentiation the initial co-ordinates (X,Y,Z) are kept unaltered.

(ii) In the material description or Lagrangian description every particle is identified by its co-ordinates at a given instant of time t_0. This is not always convenient. When we describe the flow of water in a river we do not desire to identify the location from which every particle comes. Instead we are generally interested in the instantaneous velocity field and its change with time.

In the Eulerian method a particular point in the space occupy by the fluid is selected. we denote this point by the co-ordinates (x,y,z), In this case (x,y,z) and t are independent. So, terms like $\dot{x}$, $\ddot{x}$ etc. do not occur. In this case we obtain the velocity field as a function of x,y,z and t. Thus velocity components u, v, w as a function of x, y, z and t are known.

In order to obtain the expression for accelerations in Eulerian method we assume that $u = F(x,y,z,t)$. the particle which is at (x,y,z) at time t will after a short interval of time δt move a distance $\delta x = u\delta t$

$\delta x = u\delta t$ in x direction.

$\delta y = v\delta t$ in y direction.

$\delta z = w\delta t$ in z direction.

If δu be the change in particles x-component of velocity by this time, then

$$u + \delta u = F(x+u\delta t,\ y+v\delta t,\ z+w\delta t,\ t+\delta t)$$

$$= F(x,y,z,t) + \delta t\left[u\frac{\partial F(x,y,z,t)}{\partial x} + v\frac{\partial F(x,y,z,t)}{\partial y} + w\frac{\partial F(x,y,z,t)}{\partial z} + \frac{\partial F(x,y,z,t)}{\partial t}\right] + \text{terms containing higher power of } \delta t.$$

$$= u + \delta t\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right]$$

[neglecting higher powers of δt]

Hence, the x-component of acceleration $\frac{du}{dt}$ being $\lim_{\delta t \to 0}\frac{\delta u}{\delta t}$ is equal to $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$

similarly, components of acceleration in y and z direction are

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

and $\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$ respectively.

The operator $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ is denoted by $\frac{D}{Dt}$. so acceleration components are $\frac{Du}{Dt}$, $\frac{Dv}{Dt}$ and $\frac{Dw}{Dt}$.

Stress-strain relations for an orthotropic AND for an isotropic elastic material.

The general form of Hooke's Law states that each of the components of the state of stress is linear. Mathematically

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$$
\left.
\begin{aligned}
\tau_{xx} &= c_{1} e_{xx} + c_{2} e_{yy} + c_{13} e_{zz} + c_{4} e_{yz} + c_{5} e_{zx} + c_{6} e_{xy} \\
\tau_{yy} &= c_{21} e_{xx} + c_{22} e_{yy} + c_{23} e_{zz} + c_{24} e_{yz} + c_{25} e_{zx} + c_{26} e_{xy} \\
\tau_{zz} &= c_{31} e_{xx} + c_{32} e_{yy} + c_{33} e_{zz} + c_{34} e_{yz} + c_{35} e_{zx} + c_{36} e_{xy} \\
\tau_{yz} &= c_{41} e_{xx} + c_{42} e_{yy} + c_{43} e_{zz} + c_{44} e_{yz} + c_{45} e_{zx} + c_{46} e_{xy} \\
\tau_{zx} &= c_{51} e_{xx} + c_{52} e_{yy} + c_{53} e_{zz} + c_{54} e_{yz} + c_{55} e_{zx} + c_{56} e_{xy} \\
\tau_{xy} &= c_{61} e_{xx} + c_{62} e_{yy} + c_{63} e_{zz} + c_{64} e_{yz} + c_{65} e_{zx} + c_{66} e_{xy}
\end{aligned}
\right\} \quad \text{--} (1)
$$

For monoclinic material:
[plane of symmetry one - A body has]

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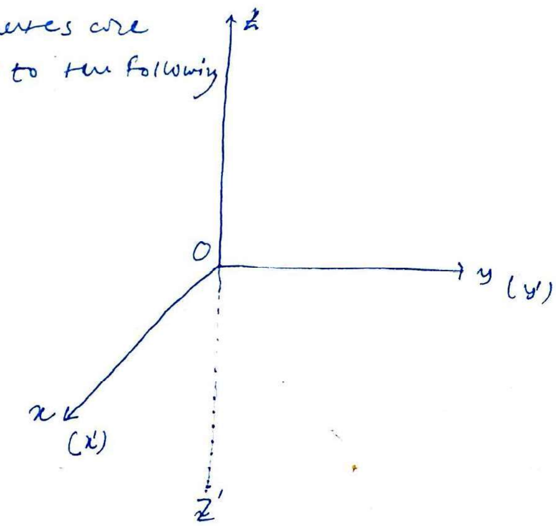
Let us take the plane of symmetry to be the plane xy-plane. The symmetry is expressed by the requirement that the elastic consts. c_{ij} do not change under a change of axes from the system ox, oy, oz to the system ox', oy', oz' where d-cs of the new axes w.r.t. the initial one are $(1,0,0)$, $(0,1,0)$, $(0,0,-1)$.

So that the co-ordinates are transformed according to the following scheme:

	x	y	z
x'	$l_1 = 1$	$m_1 = 0$	$n_1 = 0$
y'	$l_2 = 0$	$m_2 = 1$	$n_2 = 0$
z'	$l_3 = 0$	$m_3 = 0$	$n_3 = -1$



From stress-transformation law

$$\tau_{x'x'} = l_1^2 \tau_{xx} + m_1^2 \tau_{yy} + n_1^2 \tau_{zz} + 2m_1 n_1 \tau_{yz} + 2n_1 l_1 \tau_{zx} + 2 l_1 m_1 \tau_{xy}$$

$$= \tau_{xx} \quad [\text{According to the new system scheme}]$$

Similarly, $\tau_{y'y'} = \tau_{yy}$ and $\tau_{z'z'} = +\tau_{zz}$

$$\tau_{y'z'} = -\tau_{yz}, \quad \tau_{z'x'} = -\tau_{zx}, \quad \tau_{x'y'} = \tau_{xy}$$

From strain transformation law

$$e_{x'x'} = e_{xx} l_1^2 + e_{yy} m_1^2 + e_{zz} n_1^2 + 2e_{yz} m_1 n_1 + 2e_{zx} n_1 l_1 + 2e_{xy} l_1 m_1$$

So, $e_{x'x'} = e_{xx}$ [According to the new scheme] similarly, $e_{y'y'} = e_{yy}$, $e_{z'z'} = e_{zz}$

$$\left.\begin{array}{l} e_{y'z'} = -e_{yz}, \quad e_{z'x'} = -e_{zx}, \quad e_{x'y'} = e_{xy} \end{array}\right\} \cdots\cdots (2)$$

since the elastic constants are assumed not to change in the stress - strain relations in the changed co-ordinate system, so the set of eq^n (1) becomes ~~$\tau_{x'x'}$~~

$$\tau_{x'x'} = c_{11} e_{x'x'} + c_{12} e_{y'y'} + c_{13} e_{z'z'} + c_{14} e_{y'z'} + c_{15} e_{z'x'} + c_{16} e_{x'y'} \cdots (3)$$

using (2) relation (3) becomes

$$\tau_{xx} = c_{11} e_{xx} + c_{12} e_{yy} + c_{13} e_{zz} - c_{14} e_{yz} - c_{15} e_{zx} + c_{16} e_{xy}$$

Comparing this with the expression of τ_{xx} in (1)

$c_{14} = -c_{14}$ i.e. $2c_{14} = 0 \Rightarrow c_{14} = 0$

$c_{15} = -c_{15}$ i.e. $2c_{15} = 0 \Rightarrow c_{15} = 0$

similarly, considering

$\tau_{y'y'}, \tau_{z'z'}, \tau_{x'y'}$, we get

$c_{24} = c_{25} = 0$

$c_{34} = c_{35} = 0$

$c_{64} = c_{65} = 0$

Again considering $\tau_{y'z'}$, we find

$$\tau_{y'z'} = c_{41} e_{x'x'} + c_{42} e_{y'y'} + c_{43} e_{z'z'} + c_{44} e_{y'z'} + c_{45} e_{z'x'} + c_{46} e_{x'y'}$$

using (2), we get

$$\tau_{y'z'} = -\tau_{yz} = +c_{41} e_{xx} + c_{42} e_{yy} + c_{43} e_{zz} - c_{44} e_{yz} - c_{45} e_{zx} + c_{46} e_{xy}$$

or, $$\tau_{yz} = -c_{41} e_{xx} - c_{42} e_{yy} - c_{43} e_{zz} + c_{44} e_{yz} + c_{45} e_{zx} - c_{46} e_{xy}$$

Comparing this with the expression for τ_{yz} in (1), we get

$c_{41} = c_{42} = c_{43} = c_{46} = 0$

similarly, considering $\tau_{z'x'}$, we find

$c_{51} = c_{52} = c_{53} = c_{56} = 0$

so for the material with one plane of elastic - symmetry which is taken to be xy-plane, the matrix of the co-efficients of the linear forms in eq^n (1) can be written as

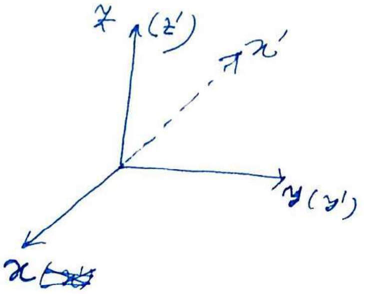
$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{21} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{31} & c_{32} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{54} & c_{55} & 0 \\ c_{61} & c_{62} & c_{63} & 0 & 0 & c_{66} \end{bmatrix} \cdots\cdots (4)$$

For orthotropic material:

Materials such as wood which possesses three mutually orthogonal planes of elastic symmetry are said to be orthopic.

Considering such material we choose the axes of co-ordinates, so that, co-ordinate planes coincide with the planes of elastic symmetry. In such case besides the symmetry of xy-plane expressed by (4), the elastic constants C_{ij} must also be such that elastic properties ~~exhibit~~ exhibit symmetry w.r.t. yz plane. Therefore C_{ij} must also be invariant under the transformation of co-ordinates defined ~~by~~ the following scheme:

	x	y	z
x'	-1	0	0
y'	0	1	0
z'	0	0	1



In this case the transformation rule of stress and strain with the help of above scheme we get

$$\tau_{x'x'} = \tau_{xx}, \quad \tau_{y'y'} = \tau_{yy}, \quad \tau_{z'z'} = \tau_{zz}, \quad \tau_{y'z'} = \tau_{yz}, \quad \tau_{z'x'} = -\tau_{zx}, \quad \tau_{x'y'} = -\tau_{xy} \quad \Big\} \cdots (5)$$

Also we have for the strains

$$e_{x'x'} = e_{xx}, \quad e_{y'y'} = e_{yy}, \quad e_{z'z'} = e_{zz}, \quad e_{y'z'} = e_{yz}, \quad e_{z'x'} = -e_{zx}, \quad e_{x'y'} = -e_{xy} \quad \Big\} \cdots (6)$$

In the new co-ordinate system since the elastic constants do not change in the stress-strain relations so ~~from (4)~~, ~~we have~~,

$$\tau_{x'x'} = C_{11} e_{x'x'} + C_{12} e_{y'y'} + C_{13} e_{z'z'} + C_{16} e_{x'y'}$$

Using (5) and (6) the above relation becomes

$$\tau_{xx} = C_{11} e_{xx} + C_{12} e_{yy} + C_{13} e_{zz} - C_{16} e_{xy}$$

Whereas from (4)

$$\tau_{xx} = C_{11} e_{xx} + C_{12} e_{yy} + C_{13} e_{zz} + C_{16} e_{xy}$$

so, $C_{16} = 0. \quad \cdots (7)$

By considering in a similar way $\tau_{y'y'}, \tau_{z'z'}, \tau_{y'z'}$ we find

$$C_{26} = C_{36} = C_{45} = 0 \quad \cdots (8)$$

Also from (4), $\tau_{zx} = C_{54} e_{yz} + C_{55} e_{zx} \quad \cdots (9)$

In the new co-ordinate system, $\tau_{z'x'} = c_{54} e_{y'z'} + c_{55} e_{z'x'}$

Using (5) and (6) we get, $-\tau_{zx} = c_{54} e_{yz} - c_{55} e_{zx}$ --- (10)

Comparing (9) and (10) we get, $c_{54} = 0$ .... (11)

Now considering $\tau_{x'y'}$ we get

$$\tau_{x'y'} = c_{61} e_{x'x'} + c_{62} e_{y'y'} + c_{63} e_{z'z'} + c_{66} e_{x'y'}$$

By the help of (5) and (6), it becomes

$$-\tau_{xy} = c_{61} e_{xx} + c_{62} e_{yy} + c_{63} e_{zz} \mp c_{66} e_{xy}$$

$$\therefore \tau_{xy} = -c_{61} e_{xx} - c_{62} e_{yy} - c_{63} e_{zz} + c_{66} e_{xy} \quad \text{--- (12)}$$

By the help of (4), we have the stress-strain relations

$$\tau_{xy} = c_{61} e_{xx} + c_{62} e_{yy} + c_{63} e_{zz} + c_{66} e_{xy} \quad \text{--- (13)}$$

Comparing (12) and (13), we get

$$c_{61} = c_{62} = c_{63} = 0 \quad \text{... (14)}$$

Taking the help of (7), (8), (11) and (14), we find that for material having elastic symmetry about xy-plane and yz-plane the matrix c_{ij} take the form

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \quad \text{...... (15)}$$

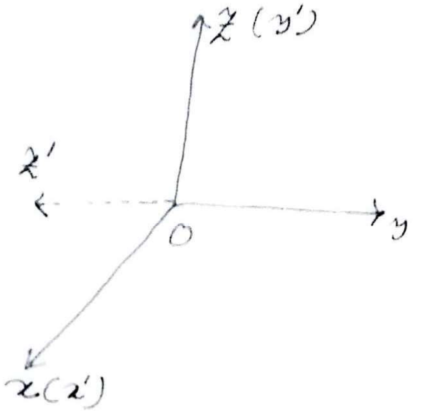
Note that Elastic symmetry in the xy and yz plane implies elastic symmetry in the xz-plane. So for the orthotropic material we have twelve constants as represented in (15). Therefore stress-strain relations for an orthotropic elastic material are given by.

$$\left.\begin{aligned} \tau_{xx} &= c_{11} e_{xx} + c_{12} e_{yy} + c_{13} e_{zz} \\ \tau_{yy} &= c_{21} e_{xx} + c_{22} e_{yy} + c_{23} e_{zz} \\ \tau_{zz} &= c_{31} e_{xx} + c_{32} e_{yy} + c_{33} e_{zz} \\ \tau_{yz} &= c_{44} e_{yz} \\ \tau_{zx} &= c_{55} e_{zx} \\ \tau_{xy} &= c_{66} e_{xy} \end{aligned}\right\} \quad \text{.... (16)}$$

For isotropic material:

From the definition of isotropic ~~material~~ medium it follows that the elastic consts. are independent of the orientation of the co-ordinate axes. The co-efficients C_{ij} remains invariant when we introduce new co-ordinate axes Ox', Oy', Oz' obtained by rotating Ox, Oy, Oz system through a right angle about Ox axis. The scheme of transformation is

	x	y	z
x'	1	0	0
y'	0	0	1
z'	0	-1	0



So from stress-transformation law

$$\left.\begin{aligned} &\tau_{x'x'} = \tau_{xx}, \quad \tau_{y'y'} = \tau_{zz}, \quad \tau_{z'z'} = \tau_{yy} \\ &\tau_{y'z'} = -\tau_{yz}, \quad \tau_{z'x'} = -\tau_{xy}, \quad \tau_{x'y'} = \tau_{zx} \\ &\text{similar formula hold for strain componants i.e.} \\ &e_{x'x'} = e_{xx}, \quad e_{y'y'} = e_{zz}, \quad e_{z'z'} = e_{yy} \\ &e_{y'z'} = -e_{yz}, \quad e_{z'x'} = -e_{xy}, \quad e_{x'y'} = e_{zx} \end{aligned}\right\} \quad \cdots (17)$$

Now from (16), $\tau_{x'x'} = C_{11} e_{x'x'} + C_{12} e_{y'y'} + C_{13} e_{z'z'}$ (on transformation)

from (17), $\tau_{xx} = C_{11} e_{xx} + C_{12} e_{zz} + C_{13} e_{yy}$ .... (18)

but from (16), $\tau_{xx} = C_{11} e_{xx} + C_{12} e_{yy} + C_{13} e_{zz}$ .... (19)

Comparing (18) and (19), we get, $C_{12} = C_{13}$ .... (20)

Again, $\tau_{y'y'} = C_{21} e_{x'x'} + C_{22} e_{y'y'} + C_{23} e_{z'z'}$

which by (17) becomes, $\tau_{zz} = C_{21} e_{xx} + C_{22} e_{zz} + C_{23} e_{yy}$

but from (16), $\tau_{zz} = C_{31} e_{xx} + C_{32} e_{yy} + C_{33} e_{zz}$

Comparing these two results, we get,

$$\left.\begin{aligned} C_{21} &= C_{31} \\ C_{23} &= C_{32} \\ C_{22} &= C_{33} \end{aligned}\right\} \quad \cdots (21)$$

It can be shown that considering transformation for $\tau_{z'z'}$ we obtain the same result (21).

Again $\tau_{y'z'} = c_{44}\, e_{y'z'}$

using (17), $\tau_{yz} = c_{44}\, e_{yz}$ which is also the result given by (16).

next $\tau_{z'x'} = c_{55}\, e_{z'x'}$, by (17) we get $\tau_{xy} = c_{55}\, e_{xy}$

but by (18) we get $\tau_{xy} = c_{66}\, e_{xy}$

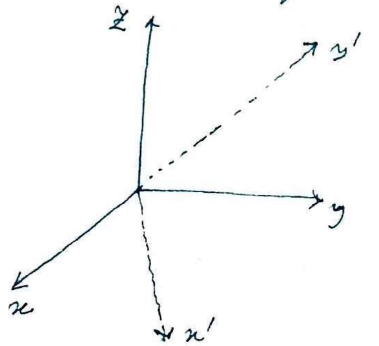
so, $c_{55} = c_{66}$ ... (22)

From $\tau_{x'y'}$ we obtain the same result (22).

similarly, a rotation of axes through a right angle about oz-axis leads to the relation

$c_{21} = c_{12}$, $c_{11} = c_{22}$, $c_{13} = c_{23}$, $c_{31} = c_{32}$, $c_{44} = c_{55}$ ... (23)

Finally we introduce the new co-ordinate system ox', oy', oz' obtain from ox, oy, oz by rotating the latter through an angle 45^0 about z-axis.



In this transformation of co-ordinates the scheme will be

	x	y	z
x'	$1/\sqrt{2}$	$1/\sqrt{2}$	0
y'	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
z'	0	0	$+1$

so from stress-transformation law

$$\left.\begin{aligned} \tau_{x'y'} &= -\tfrac{1}{2}\tau_{xx} + \tfrac{1}{2}\tau_{yy} \\ \text{also, } e_{x'y'} &= -\tfrac{1}{2} e_{xx} + \tfrac{1}{2} e_{yy} \end{aligned}\right\} \cdots (24)$$

From (16), $\tau_{x'y'} = c_{66}\, e_{x'y'}$

using (24), $\tau_{xx} - \tau_{yy} = c_{66}(e_{xx} - e_{yy})$ ... (25)

By (18) $\tau_{xx} = c_{11} e_{xx} + c_{12} e_{yy} + c_{13} e_{zz}$

$\tau_{yy} = c_{21} e_{xx} + c_{22} e_{yy} + c_{23} e_{zz}$

using the results that $c_{11} = c_{22}$ and $c_{12} = c_{13} = c_{23} = c_{21}$

we have $\tau_{xx} - \tau_{yy} = (c_{11} - c_{12})(e_{xx} - e_{yy})$ ... (26)

From (25) and (26), we get

$c_{11} - c_{12} = c_{66} = 2\mu$ (say) .... (27)

Let us put, $c_{12} = c_{13} = c_{23} = c_{21} = c_{31} = c_{32} = -1$

$\therefore c_{11} = 2\mu + \lambda = c_{22} = c_{33}$

$\therefore$ by (16), $\tau_{xx} = c_{11} e_{xx} + c_{12} e_{yy} + c_{13} e_{zz}$

$= (\lambda + 2\mu) e_{xx} - \lambda . e_{yy} - \lambda . e_{zz}$

$= \lambda . \theta + 2\mu e_{xx}$, $\quad \theta = e_{xx} + e_{yy} + e_{zz}$

$\tau_{yy} = c_{21} e_{xx} + c_{22} e_{yy} + c_{23} e_{zz}$

$= \lambda . e_{xx} + (\lambda + 2\mu) e_{yy} - \lambda . e_{zz}$

$= \lambda . \theta + 2\mu e_{yy}$

similarly, $\tau_{zz} = \lambda . \theta + 2\mu e_{zz}$

and $\tau_{yz} = c_{44} e_{yz} = 2\mu e_{yz}$

$\tau_{zx} = c_{55} e_{zx} = 2\mu e_{zx}$

$\tau_{xy} = c_{66} e_{xy} = 2\mu e_{xy}$

thus the generalised Hooke's law for a homogeneous isotropic body can be written in the form

$\tau_{ij} = \lambda\theta\, \delta_{ij} + 2\mu e_{ij}$ where $i, j = 1, 2, 3$

$\theta = e_{ii}$

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$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

principal axes of stress and strain coincide at every point of an isotropic elastic body:

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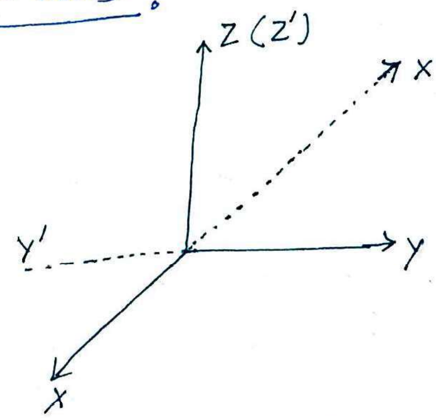


Let OX, OY, OZ be the principal axes of strain at any point O of an isotropic elastic body. So with reference to these set of axes

$e_{yz} = e_{zx} = e_{xy} = 0$.

we shall prove that OX, OY, OZ are also the principal axes of stress at the point O of the body i.e. we shall prove that $\tau_{yz} = \tau_{zx} = \tau_{xy} = 0$

By generalised Hooke's law we have

$\tau_{yz} = A e_{xx} + B e_{yy} + C e_{zz}$ ----- (1) where A, B, C are constants.

Let OX', OY', OZ' be the new set of rectangular axes at O obtain by rotating the system of axes about OZ through an angle $180°$.

If (x, y, z) and (x', y', z') be the co-ordinates at any point referred to the system of axes then the scheme of transformation is given by

	x	y	z
x'	-1	0	0
y'	0	-1	0
z'	0	0	1

Now since the medium is isotropic the same formula ① will hold when refer to dashed system of co-ordinate

i.e. $\tau_{y'z'} = A e_{x'x'} + B e_{y'y'} + C e_{z'z'}$ --- ②

But $e_{x'x'} = l_1^2 e_{xx} + m_1^2 e_{yy} + n_1^2 e_{zz} + 2l_1 m_1 e_{xy} + 2m_1 n_1 e_{yz} + 2n_1 l_1 e_{zx}$

putting $l_1 = -1, m_1 = 0, n_1 = 0$, we get

$e_{x'x'} = e_{xx}$ similarly $e_{y'y'} = e_{yy}$, $e_{z'z'} = e_{zz}$

But $\tau_{y'z'} = l_2 l_3 \tau_{xx} + m_2 m_3 \tau_{yy} + n_2 n_3 \tau_{zz} + (l_2 m_3 + l_3 m_2)\tau_{xy} + (m_2 n_3 + m_3 n_2)\tau_{yz} + (l_2 n_3 + l_3 n_2)\tau_{zx}$

putting $l_2 = 0, m_2 = -1, n_2 = 0$

$l_3 = 0, m_3 = 0, n_3 = 1$

we have, $\tau_{y'z'} = -\tau_{yz}$

so from ②, $-\tau_{yz} = A e_{xx} + B e_{yy} + C e_{zz}$

$\tau_{yz} = -A e_{xx} - B e_{yy} - C e_{zz}$ ---- ③

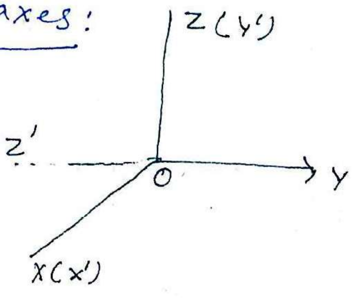
Hence from ① and ③ $A = B = C = 0$

$\therefore \tau_{yz} = 0$

similarly we can shown that, $\tau_{zx} = \tau_{xy} = 0$

$\therefore$ ox, oy, oz are the principal axes of stress showing in an isotropic body principal axes of stress and strain coincide.

Stress-strain relation for an isotropic elastic body subject to principal axes:



Let ox, oy, oz be the principal axes of stress-strain at any point O of an isotropic elastic body. So by generalised Hooke's law we may write

$$\tau_{xx} = a e_{xx} + b e_{yy} + c e_{zz} \quad \cdots \text{(1)}$$

Let OX', OY', OZ' be the new set of rectangular axes obtain by rotating the system $OXYZ$ through an angle $90°$ about OX.

Since the medium is isotropic the formula (1) will hold when referred to new system of axes OX', OY', OZ' i.e.

$$\tau_{x'x'} = a e_{x'x'} + b e_{y'y'} + c e_{z'z'} \quad \cdots \text{(2)}$$

If (x, y, z) and (x', y', z') be the co-ordinates of any point referred to $OXYZ$ and $OX'Y'Z'$ then the scheme of transformation is

	X	Y	Z
X'	1	0	0
Y'	0	0	1
Z'	0	-1	0

So $e_{z'z'} = l_3^2 e_{xx} + m_3^2 e_{yy} + n_3^2 e_{zz} + 2l_3m_3 e_{xy} + 2m_3n_3 e_{yz} + 2l_3n_3 e_{zx}$

Now by transformation law $e_{z'z'} = e_{yy}$

Similarly, $e_{x'x'} = e_{xx}$, $e_{y'y'} = e_{zz}$

Also $\tau_{x'x'} = \tau_{xx}$

So from (2) $\tau_{xx} = a e_{xx} + b e_{zz} + c e_{yy} \quad \cdots \text{(3)}$

From (1) and (3) we have $b = c$

So, $\tau_{xx} = a e_{xx} + b(e_{yy} + e_{zz})$

$= b(e_{xx} + e_{yy} + e_{zz}) + (a-b) e_{xx}$

$= b\theta + (a-b) e_{xx}$ where $\theta = e_{xx} + e_{yy} + e_{zz}$

putting $b = +\lambda$ and $a - b = 2\mu$

we have, $\tau_{xx} = +\lambda\theta + 2\mu e_{xx} \quad \cdots \text{(4a)}$

similarly, $\tau_{yy} = +\lambda\theta + 2\mu e_{yy} \quad \cdots \text{(4b)}$

$\tau_{zz} = +\lambda\theta + 2\mu e_{zz} \quad \cdots \text{(4c)}$

In order to get the relation of $\tau_{xy}, \tau_{yz}, \tau_{zx}$ with strain components we multiply (4a), (4b) and (4c) by x^2, y^2 and z^2 respectively and adding we get

$$\tau_{xx}x^2 + \tau_{yy}y^2 + \tau_{zz}z^2 = +\lambda\theta(x^2+y^2+z^2) + 2\mu(e_{xx}x^2 + e_{yy}y^2 + e_{zz}z^2) \quad \cdots \text{(5)}$$