$\Rightarrow \quad R = c_1 \ln r + c_2$

also $H'' = 0 \Rightarrow H = c_3\theta + c_4$

$\therefore \; u(r,\theta) = (c_1 \ln r + c_2)(c_3\theta + c_4)$ ---- (iv)

Now for the interior problem $r = 0$ is a point in the domain R and since $\ln r$ is not defined at $r = 0$, so the solutions (iii) and (iv) are not acceptable. Then the required solution is---

$$u(r,\theta) = (c_1 r^{\lambda} + c_2 r^{-\lambda})(c_3 \cos\lambda\theta + c_4 \sin\lambda\theta)$$

Now, ~~fo~~ from the periodicity condition of θ we get.

$$u(r,\theta) = u(r, \theta + 2\pi)$$

$\Rightarrow \quad c_3 \cos\lambda\theta + c_4 \sin\lambda\theta = c_3 \cos\lambda(\theta + 2\pi) + c_4 \sin\lambda(\theta + 2\pi)$

$\Rightarrow \quad c_3 [\cos\lambda\theta - \cos(\lambda\theta + 2\pi\lambda)] + c_4 [\sin\lambda\theta - \sin(\lambda\theta + 2\pi\lambda)] = 0$

$\Rightarrow \quad c_3 \cdot 2\sin(\lambda\theta + \lambda\pi)\sin\lambda\pi - c_4 \cdot 2\cos(\lambda\theta + \lambda\pi)\sin\lambda\pi = 0$

$\Rightarrow \quad 2\sin\lambda\pi \, [c_3 \sin(\lambda\theta + \lambda\pi) - c_4 \cos(\lambda\theta + \lambda\pi)] = 0$

$\Rightarrow \quad \sin\lambda\pi = 0 = \sin n\pi$

$\Rightarrow \quad \lambda = n, \quad n = 0, 1, 2 \ldots$

$\therefore$ By ~~pr~~ the principal of superposition we get..

$$u(r,\theta) = \sum_{n=0}^{\infty} (c_n r^n + d_n r^{-n})(a_n \cos n\theta + b_n \sin n\theta)$$

At $r = 0$, the solution should be finite which requies $d_n = 0$ thus..

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

for $n = 0$. let the constant ~~to~~ A_0 be $A_0/2$, then the solution is..

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \quad \text{---- (v)}$$

which is a full-range Fourier-series...

$$u(a,\theta) = f(\theta).$$

$$f(\theta) = \sum_{n=0}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$a^n A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$$

$$a^n B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$$

replacing θ by ϕ in the above equations and putting in (v) we get.

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \sum_{n=1}^{\infty}\left[\frac{r^n}{a^n}\cdot\frac{\cos n\theta}{\pi}\int_0^{2\pi}\cos(n\phi)\,f(\phi)\,d\phi + \frac{r^n}{a^n}\cdot\frac{\sin n\theta}{\pi}\int_0^{2\pi}\sin n\phi\, f(\phi)\,d\phi\right]$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta)\,d\phi$$

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta)\right]d\phi \quad \text{--- (vi)}$$

let $$C = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta)$$

$$S = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \sin n(\phi-\theta)$$

$$C+iS = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\phi-\theta)}$$

$$= \sum_{n=1}^{\infty}\left[\frac{r}{a} e^{i(\phi-\theta)}\right]^n$$

since $r<a$, so $r/a < 1$ and $|e^{i(\phi-\theta)}| < 1$

$$\therefore\ C+iS = \sum_{n=1}^{\infty}\left[\left\{\frac{r}{a} e^{i(\phi-\theta)}\right]^n$$

$$= \frac{r/a\ e^{i(\phi-\theta)}}{1 - r/a\ e^{i(\phi-\theta)}}$$

$$= \frac{r/a\left\{e^{i(\phi-\theta)} - r/a\right\}}{\left[1-\frac{r}{a}e^{i(\phi-\theta)}\right]\left[1-\frac{r}{a}e^{-i(\phi-\theta)}\right]}$$

Equating the real part on both sides, we get

$$c = \frac{r/a \cos(\phi-\theta) - r^2/a^2}{1 - 2r/a \cos(\phi-\theta) + r^2/a^2}$$

$$\therefore \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\phi-\theta) = \frac{1}{2} + \frac{ar\cos(\phi-\theta) - r^2}{a^2 - 2ar\cos(\phi-\theta) + r^2}$$

$$= \frac{a^2 - 2ar\cos(\phi-\theta) + r^2 + 2ar\cos(\phi-\theta) - 2r^2}{2(a^2 - 2ar\cos(\phi-\theta) + r^2)}$$

$$= \frac{a^2 - r^2}{2(a^2 - 2ar\cos(\phi-\theta) + r^2)}$$

Thus the required solution takes the form.

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2) f(\phi)}{[a^2 - 2ar\cos(\phi-\theta) + r^2]} \, d\phi$$

which gives a unique solution for the Dirichlet problem.

2016

- **Exterior Dirichlet problem for a circle :-**

The exterior Dirichlet problem is described by.

$$\nabla^2 u = 0$$

$$\text{B.C} \quad u(a,\theta) = f(\theta)$$

u must be bounded as $r \to \infty$

Now, the equation $\nabla^2 u = 0$ in polar co-ordinates can be written as.

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{---- (i)}$$

Let the solution be. $u(r,\theta) = R(r).H(\theta)$

putting this in (i) we get ---

$$R''H + \frac{1}{r} R'H + \frac{1}{r^2} RH'' = 0$$

$$\Rightarrow \frac{r^2 R'' + rR'}{R} = -\frac{H''}{H} = k$$

$$\Rightarrow r^2 R'' + rR' - kR = 0$$

$$H'' + kH = 0$$

— $n = n$ then

$$r^2 R'' + rR' - \lambda^2 R = 0$$

which is a Euler-type of equation and can be solved by putting $r = e^z$

The solution is. $R = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$

$$\Rightarrow R = c_1 r^\lambda + c_2 r^{-\lambda}$$

$H'' + \lambda^2 H = 0 \Rightarrow H = c_3 \cos\lambda\theta + c_4 \sin\lambda\theta$

$\therefore u(r,\theta) = (c_1 r^\lambda + c_2 r^{-\lambda})(c_3 \cos\lambda\theta + c_4 \sin\lambda\theta) \ldots$ (ii)

Case - II :-

Let $k = -\lambda^2$. then

$r^2 R'' + rR' + \lambda^2 R = 0$ and $H'' - \lambda^2 H = 0$

The solutions are ..

$$R = c_1 \cos(\lambda \ln r) + c_2 \sin(\lambda \ln r)$$

$$H = c_3 e^{\lambda\theta} + c_4 e^{-\lambda\theta}$$

$u(r,\theta) = (c_1 \cos(\lambda \ln r) + c_2 \sin(\lambda \ln r))(c_3 e^{\lambda\theta} + c_4 e^{-\lambda\theta})$ --- (iii)

Case - III :- Let $k = 0$. then

$$r^2 R'' + rR' = 0$$

put $R' = V$. then

$$r^2 \frac{dV}{dr} + rV = 0$$

$$\frac{dV}{V} + \frac{dr}{r} = 0$$

integrating $\ln Vr = \ln c_1 \Rightarrow$

$$V = \frac{c_1}{r} = \frac{dR}{dr}$$

$$\Rightarrow R = c_1 \ln r + c_2$$

also $H'' = 0 \Rightarrow H = c_3\theta + c_4$

$\therefore u(r,\theta) = (c_1 \ln r + c_2)(c_3\theta + c_4) \ldots$ (iv)

Now, ~~unsmet~~ as $r \to \infty$. $\ln r$ is not defined.

so the solutions (iii) and (iv) are not acceptable.

then the required solution is.

$$u(r,\theta) = (c_1 r^\lambda + c_2 r^{-\lambda})(c_3 \cos\lambda\theta + c_4 \sin\lambda\theta)$$

Now. ~~for~~ from the periodicity condition of θ, we get.

$$u(r,\theta) = u(r, \theta+2\pi)$$

$$\Rightarrow \quad c_3 \cos\lambda\theta + c_4 \sin\lambda\theta = c_3 \cos\lambda(\theta+2\pi) + c_4 \sin\lambda(\theta+2\pi)$$

$$\Rightarrow \quad c_3 \, 2\sin(\lambda\theta+\lambda\pi).\sin\lambda\pi - c_4 \, 2\cos(\lambda\theta+\lambda\pi)\sin\lambda\pi = 0$$

$$\Rightarrow \quad 2\sin\lambda\pi \left[c_3 \sin(\lambda\theta+\lambda\pi) - c_4 \cos(\lambda\theta+\lambda\pi)\right] = 0$$

$$\Rightarrow \quad \sin\lambda\pi = 0 = \sin n\pi$$

$$\Rightarrow \quad \lambda = n \quad , \quad n = 0, 1, 2 \ldots$$

$\therefore$ By the principal of superposition we get..

$$u(r,\theta) = \sum_{n=0}^{\infty} (c_n r^n + d_n r^{-n})(a_n \cos n\theta + b_n \sin n\theta)$$

Now, as $r \to \infty$, we require u to be bounded, so $c_n = 0$

$$\therefore \quad u(r,\theta) = \sum_{n=0}^{\infty} r^{-n} (A_n \cos n\theta + B_n \sin n\theta)$$

for $n=0$ let the constant A_0 be $\frac{A_0}{2}$, then the solution is..

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta) \text{ --- (v)}$$

Now, using BC. $u(a,\theta) = f(\theta)$, we get..

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta + B_n \sin n\theta)$$

which is a full-range Fourier series in $f(\theta)$, where..

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$a^{-n} A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta$$

$$a^{-n} B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta$$

Now, replacing θ by ϕ in the above equations and putting in (v) we get.

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \sum_{n=1}^{\infty}\Big[\frac{r^{-n}a^n}{\pi}\cos n\theta \int_0^{2\pi}\cos n\phi \, f(\phi)\,d\phi$$
$$+ \frac{r^{-n}a^n}{\pi}\sin n\theta \int_0^{2\pi} \sin n\phi \, f(\phi)\,d\phi$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\phi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \cos n(\phi-\theta)\right]d\phi \;\ldots\; \text{(vi)}$$

Let $\quad C = \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \cos n(\phi-\theta)$

$$S = \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \sin n(\phi-\theta)$$

Then $\quad C + iS = \sum_{n=1}^{\infty} \left[\frac{a}{r} e^{i(\phi-\theta)}\right]^n$

since $\frac{a}{r} < 1$, $|e^{i(\phi-\theta)}| \le 1$, we have.

$$C + iS = \frac{a}{r} \frac{e^{i(\phi-\theta)}}{1 - \frac{a}{r} e^{i(\phi-\theta)}}$$

$$= \frac{a/r \; e^{i(\phi-\theta)} \left[1 - \frac{a}{r} e^{-i(\phi-\theta)}\right]}{\left[1 - \frac{a}{r} e^{i(\phi-\theta)}\right]\left[1 - \frac{a}{r} e^{-i(\phi-\theta)}\right]}$$

$$= \frac{a/r \left(e^{i(\phi-\theta)} - \frac{a}{r}\right)}{1 - \frac{2a}{r}\cos(\phi-\theta) + \frac{a^2}{r^2}}$$

equating the real co-efficients from both sides...

$$C = \frac{a/r \cos(\phi-\theta) - a^2/r^2}{1 - \frac{2a}{r}\cos(\phi-\theta) + a^2/r^2}$$

$$= \frac{ar\cos(\phi-\theta) - a^2}{r^2 - 2ar\cos(\phi-\theta) + a^2}$$

$$\therefore \frac{1}{2} + \sum_{n=1}^{\infty} \cancel{\frac{r}{a}} \left(\frac{a}{r}\right)^n \cos n(\phi-\theta) = \frac{1}{2} + \frac{ar\cos(\phi-\theta) - a^2}{r^2 - 2ar\cos(\phi-\theta) + a^2}$$

$$= \frac{r^2 - 2ar\cos(\phi-a) + a^2 + 2ar\cos(\phi-\theta) - 2a^2}{2(\cancel{\phi}\, r^2 - 2ar\cos(\phi-\theta) + a^2)}$$

$$= \frac{r^2 - a^2}{2(r^2 - 2ar\cos(\phi-\theta) + a^2)}$$

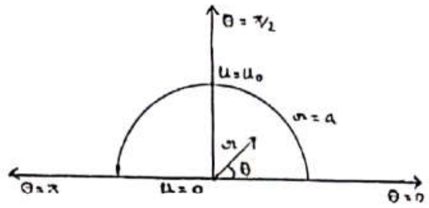
~~∴~~ the required solution takes the form:-

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(r^2 - a^2) f(\phi)}{[r^2 - 2ar\cos(\phi-\theta) + a^2]} \, d\phi$$

- ~~Find the steady~~ state temperature distribution in a semi-circular plate of radius a, ~~insulated on both~~ the faces with its curved boundary kept at a constant ~~temperature u_0~~ and its bounding diameter kept at zero temperature.

Scanned by CamScanner

$\Rightarrow$



The problem can be stated as follows..

$$\nabla^2 u(r,\theta) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{--- (i)}$$

$$\text{B.C: } u(a,\theta) = u_0 \ , \quad u(r,0) = 0, \quad u(r,\pi) = 0$$

Let the solution be -- $u(r,\theta) = R(r).H(\theta)$

putting this in (i) we get..

$$R''H + \frac{1}{r}R'H + \frac{1}{r^2}RH'' = 0$$

$$\Rightarrow \frac{r^2R'' + rR'}{R} = -\frac{H''}{H} = k$$

$$\Rightarrow r^2R'' + rR' - kR = 0$$

$$H'' + kH = 0$$

case - I :- when ~~$k>0\ (=\lambda^2)$, then~~ $k<0\ (=-\lambda^2)$, then

$$r^2R'' + rR' + \lambda^2 R = 0 \quad \text{and} \quad H'' - \lambda^2 H = 0$$

The solutions are...

$$R = c_1 \cos(\lambda \ln r) + c_2 \sin(\lambda \ln r)$$

$$H = c_3 e^{\lambda\theta} + c_4 e^{-\lambda\theta}.$$

$$\therefore u(r,\theta) = (c_1\cos(\lambda\ln r) + c_2\sin(\lambda\ln r))(c_3e^{\lambda\theta} + c_4e^{-\lambda\theta})$$

Now, at the point $r=0$, $\ln r$ is not defined, so the solution is not acceptable.

case - II :- when $k=0$ then

$$r^2R'' + rR' = 0 \Rightarrow R = c_1 \ln r + c_2$$

$$H'' = 0 \Rightarrow H = c_3\theta + c_4$$

$$\therefore u(r,\theta) = (c_1\ln r + c_2)(c_3\theta + c_4)$$

Similarly at $r=0$, $\ln r$ is not defined, so this solution is also not acceptable.

case - III :- when ~~$k<0$~~ $k>0\ (=\lambda^2)$, then

$$r^2R'' + rR' - \lambda^2R = 0$$

hich is a Euler-type of equation and can be solved by putting $r = e^z$

the solution is -- $R = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$

$$\Rightarrow R = c_1 r^{\lambda} + c_2 r^{-\lambda}$$

$H'' + \lambda^2 H = 0 \Rightarrow H = c_3 \cos\lambda\theta + c_4 \sin\lambda\theta$

$$u(r,\theta) = (c_1 r^{\lambda} + c_2 r^{-\lambda})(c_3 \cos\lambda\theta + c_4 \sin\lambda\theta)$$

Now using BC. $u(r,0) = u(r,\pi) = 0$ we get - -

$$c_3 = 0$$

$$c_4 \sin\lambda\pi = 0$$

$$\Rightarrow \sin\lambda\pi = 0 = \sin n\pi \quad (\text{for a non-trivial solution } c_4 \neq 0)$$

$$\Rightarrow \lambda = n, \quad n = 1, 2 \ldots$$

~~$u(r,\theta) =$~~

Also we observe that as $r \to 0$, the term $r^{-\lambda} \to \infty$, but the solution should be finite at $r = 0$. so $c_2 = 0$

Now by the principal of superposition we get - - -

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta$$

using B C. $u(a,\theta) = u_0$ we get

$$u_0 = \sum_{n=1}^{\infty} A_n a^n \sin n\theta$$

which is a half-range Fourier series.

$$A_n a^n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin n\theta \, d\theta = \frac{2u_0}{\pi}\frac{1}{n}[1-(-1)^n]$$

$$= \frac{4u_0}{n\pi}, \quad n = 1, 3 \ldots$$

$$= 0 \quad n = 2, 4 \ldots$$

$$A_n = \frac{4u_0}{n\pi a^n}$$

$\therefore$ The required general solution is.

$$u(r,\theta) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^n \sin n\theta$$

Interior Neumann problem for a circle :-

The interior Neumann problem for a circle is described by. -

$$\nabla^2 u = 0 \ , \quad 0 \le r < a \ , \quad 0 \le \theta \le 2\pi$$

$$B.C. \quad \frac{\partial u}{\partial r} = \frac{\partial u(a,\theta)}{\partial r} = g(\theta) \ , \quad r = a$$

Let the solution be $u(r,\theta) = R(r) . H(\theta)$

putting this in $\nabla^2 u(r,\theta) = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$, we get . .

$$R''H + \frac{1}{r} R'H + \frac{1}{r^2} RH'' = 0$$

$$\Rightarrow \frac{r^2 R'' + rR'}{R} = -\frac{H''}{H} = k$$

$$\Rightarrow r^2 R'' + rR' - kR = 0$$

$$H'' + kH = 0$$

case - I :- when $k < 0 \ (= -\lambda^2)$, then

$$r^2 R'' + rR' + \lambda^2 R = 0 \Rightarrow R = c_1 \cos(\lambda \ln r) + c_2 \sin(\lambda \ln r)$$

$$H'' - \lambda^2 H = 0 \Rightarrow H = c_3 e^{\lambda\theta} + c_4 e^{-\lambda\theta}$$

$$\therefore u(r,\theta) = (c_1 \cos(\lambda \ln r) + c_2 \sin(\lambda \ln r))(c_3 e^{\lambda\theta} + c_4 e^{-\lambda\theta})$$

Now, at the point $r = 0$, $\ln r$ is not defined, so the solution is not acceptable.

case - II :- when $k = 0$, then

$$r^2 R'' + rR' = 0 \Rightarrow R = c_1 \ln r + c_2$$

$$H'' = 0 \Rightarrow H = c_3 \theta + c_4$$

$$\therefore u(r,\theta) = (c_1 \ln r + c_2)(c_3 \theta + c_4)$$

similarly at $r = 0$, $\ln r$ is not defined, so, this solution is also not acceptable.

case - III :- when $k > 0 \ (= \lambda^2)$, then . .

$$r^2 R'' + rR' - \lambda^2 R = 0$$

which is a Euler type of equation and can be solved by putting $r = e^z$.

The solution is . . $R = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$

$$\Rightarrow R = c_1 r^{\lambda} + c_2 r^{-\lambda}$$

$-\lambda^2 H = 0 \Rightarrow H = c_3 \cos\lambda\theta + c_4 \sin\lambda\theta$

$u(r,\theta) = (c_1 r^{\lambda} + c_2 r^{-\lambda})(c_3 \cos\lambda\theta + c_4 \sin\lambda\theta)$

from the periodicity condition of θ, we get..

$$u(r,\theta) = u(r, \theta + 2\pi)$$

$\Rightarrow c_3 \cos\lambda\theta + c_4 \sin\lambda\theta = c_3 \cos\lambda(\theta+2\pi) + c_4 \sin\lambda(\theta+2\pi)$

$\Rightarrow c_3 [\cos\lambda\theta - \cos(\lambda\theta + 2\pi\lambda)] + c_4 [\sin\lambda\theta - \sin(\lambda\theta + 2\pi\lambda)] = 0$

$\Rightarrow c_3\, 2\sin(\lambda\theta+\lambda\pi)\sin\lambda\pi - c_4\, 2\cos(\lambda\theta+\lambda\pi)\sin\lambda\pi = 0$

$\Rightarrow \sin\lambda\pi\,[2c_3 \sin(\lambda\theta+\lambda\pi) - 2c_4\cos(\lambda\theta+\lambda\pi)] = 0$

$\Rightarrow \sin\lambda\pi = 0 = \sin n\pi$

$\Rightarrow \lambda = n, \quad n = 0, 1, 2 \ldots$

By the principal of superposition we get..

$$u(r,\theta) = \sum_{n=0}^{\infty} (c_n r^n + d_n r^{-n})(a_n \cos n\theta + b_n \sin n\theta)$$

At $r=0$, the solution should be finite so $d_n = 0$

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

for $n=0$, let the constant A_0 be $A_0/2$, then the solution is ...

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \quad \text{.....} (i)$$

$$\therefore \frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} n r^{n-1} (A_n \cos n\theta + B_n \sin n\theta)$$

using BC ~~$\frac{\partial u}{\partial r}(a,\theta) = g(\theta)$, we get~~. $\frac{\partial u}{\partial r}(a,\theta) = g(\theta)$

$$g(\theta) = \sum_{n=1}^{\infty} n a^{n-1} (A_n \cos n\theta + B_n \sin n\theta)$$

which is a full-range Fourier series in $g(\theta)$, where

$$n a^{n-1} A_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta \, d\theta$$

$$n a^{n-1} B_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta \, d\theta$$

Now we replace θ by ϕ in the equations and putting in (i) we get.

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{n\pi a^{n-1}} \int_0^{2\pi} g(\phi)\,(\cos n\phi \cos n\theta + \sin n\phi \sin n\theta)\,d\phi$$

$$= \frac{A_0}{2} + \int_0^{2\pi} g(\phi) \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{a}{n\pi} \cos n(\phi-\theta)\,d\phi$$

Let $c = \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{a}{n\pi} \cos n(\phi-\theta)$

$$S = \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{a}{n\pi} \sin n(\phi-\theta)$$

$$\therefore c + iS = \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\phi-\theta)} \frac{a}{n\pi}$$

$$= \frac{a}{\pi} \sum_{n=1}^{\infty} \left[\frac{r}{a} e^{i(\phi-\theta)}\right]^n \frac{1}{n}$$

$$= \frac{a}{\pi} \left[\frac{\frac{r}{a} e^{i(\phi-\theta)}}{1} + \frac{\left(\frac{r}{a} e^{i(\phi-\theta)}\right)^2}{2} + \frac{\left(\frac{r}{a} e^{i(\phi-\theta)}\right)^3}{3} + \cdots \right]$$

$$= -\frac{a}{\pi} \ln\left[1 - \frac{r}{a} e^{i(\phi-\theta)}\right]$$

$$= -\frac{a}{\pi} \ln\left[1 - \frac{r}{a}\cos(\phi-\theta) - i\frac{r}{a}\sin(\phi-\theta)\right]$$

To get the real part of $\ln z$; let.

$$W = \ln z \Rightarrow z = e^W$$

i.e. $x + iy = e^{u+iv} = e^u \cos v + i e^u \sin v$

$$\therefore e^{2u} = x^2 + y^2 = |z|^2$$

$$\therefore u = \ln|z|$$

$$\therefore c = -\frac{a}{\pi} \ln\sqrt{\left(1 - \frac{r}{a}\cos(\phi-\theta)\right)^2 + \left(\frac{r}{a}\sin(\phi-\theta)\right)^2}$$

$$= -\frac{a}{\pi} \ln\sqrt{\frac{a^2 - 2ar\cos(\phi-\theta) + r^2}{a^2}}$$

$\therefore$ The required solution is...

$$u(r,\theta) = \frac{A_0}{2} - \frac{a}{\pi} \int_0^{2\pi} \ln\sqrt{\frac{a^2 - 2ar\cos(\phi-\theta) + r^2}{a^2}}\; g(\phi)\,d\phi$$

Solution of Laplace equation in cylindrical co-ordinates :-

The Laplace equation in cylindrical co-ordinates is of the form.

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = 0 \quad \text{---} (i)$$

Let the solution of (i) be.

$$u(r, \theta, z) = F(r, \theta) . Z(z)$$

putting this in (i) we get -

$$\frac{\partial^2 F}{\partial r^2} Z + \frac{1}{r} \frac{\partial F}{\partial r} Z + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} Z + F \frac{\partial^2 Z}{\partial z^2} = 0$$

$$\Rightarrow \left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} \right) \frac{1}{F} = - \frac{\partial^2 Z}{\partial z^2} . \frac{1}{Z} = k \text{ (say)}$$

$$\therefore \frac{d^2 Z}{dz^2} + kZ = 0 \quad \text{--} (ii) \quad \text{and} \quad \frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} - kF = 0 \quad \text{---} (iii)$$

If $k < 0$ then the solution of (ii) is .. $Z = c_1 e^{\sqrt{k} z} + c_2 e^{-\sqrt{k} z}$

If $k > 0$, then the solution of (ii) is --- $Z = c_1 \cos \sqrt{k} z + c_2 \sin \sqrt{k} z$

If $k = 0$, then the solution of (ii) is --- $Z = c_1 z + c_2$

From physical considerations, one would expect a solution which decays with increasing z, so the solution to negative k is acceptable. Let $k = -\lambda^2$, then

$$Z = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$$

Equation (iii) becomes -

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} + \lambda^2 F = 0 \quad \dots (iv)$$

Let $F(r, \theta) = f(r) . H(\theta)$

putting this in (iv), we get --

$$f'' H + \frac{1}{r} f' H + \frac{1}{r^2} f H'' + \lambda^2 f H = 0$$

$$\Rightarrow (r^2 f'' + r f' + \lambda^2 r^2 f) . \frac{1}{f} = - \frac{H''}{H} = k'$$

$$\Rightarrow r^2 f'' + r f' + (\lambda^2 r^2 - k') f = 0$$

$$H'' + k' H = 0$$

From physical consideration, we expect the solution to be periodic in θ, which can be obtained when k' is +ve. $\therefore$ $k' = n^2$

$$\therefore \quad H = c_3 \cos n\theta + c_4 \sin n\theta$$

for $k' = n^2$, we have..

$$r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} + (\lambda^2 r^2 - n^2) f = 0$$

which is a Bessel's equation, whose general solution is given by..

$$f = A J_n(\lambda r) + B Y_n(\lambda r)$$

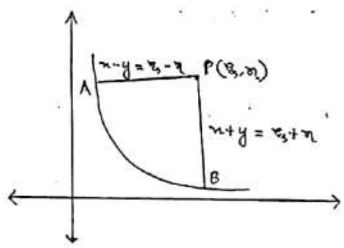
Here $J_n(\lambda r)$ and $Y_n(\lambda r)$ are the n-th order Bessel function of first and 2nd kind, respectively.

But $Y_n(\lambda r) \to \infty$ as $r \to 0$, so for a ~~bounded~~ finite solution $B = 0$

Hence the required general solution is —

$$u(r, \theta, z) = J_n(\lambda r) \left(c_1 e^{\lambda z} + c_2 e^{-\lambda z}\right)\left(c_3 \cos n\theta + c_4 \sin n\theta\right)$$

- Riemann - Volterra method for solving the Cauchy problem for one-dimensional wave equation :-



$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} \quad \cdots (i)$$ when z, z_x, z_y are described along a curve C in the xy-plane

comparing (i) ~~and~~ with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ we get.

$$R = 1, \quad S = 0, \quad T = -1$$

Then the λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to ...

$$\lambda^2 - 1 = 0$$

$$\Rightarrow \lambda = \pm 1$$

Then the corresponding Characteristic equations are —

$$\left.\begin{aligned} dy/dx + 1 = 0 &\Rightarrow x + y = c_1 \\ dy/dx - 1 = 0 &\Rightarrow x - y = c_2 \end{aligned}\right\} \text{— (ii)}$$

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Let $P(\xi, \eta)$ be any point in xy-plane, we now obtain characteristic of (i) passing through P.

So. putting $x = \xi$ and $y = \eta$ in equation (ii) we have..

$$c_1 = \xi + \eta, \qquad c_2 = \xi - \eta$$

Hence the characteristic of (i) passing through the point P are given by..

$$x + y = \xi + \eta$$

$$x - y = \xi - \eta$$

which are shown by the ~~set~~ straight line PB and PA respectively.

Here the characteristic PA and PB cut the given curve in A and B respectively.

let c' denotes the closed curve... c': PABP which is made up of straight line PA, curve c and straight line BP

Let S be the area enclose by c'

Equation (i) can be written as..

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{--- (iii)}$$

integrating both sides of (iii) over S. we get--

$$\iint_S \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) dx\,dy = 0$$

$$\Rightarrow \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\,dy = 0$$

$$\Rightarrow \oint_{c'} \left(\frac{\partial z}{\partial y}\,dx + \frac{\partial z}{\partial x}\,dy\right) = 0$$

$$\Rightarrow \oint_{AB} \left(\frac{\partial z}{\partial y}\,dx + \frac{\partial z}{\partial x}\,dy\right) + \oint_{BP} \left(\frac{\partial z}{\partial y}\,dx + \frac{\partial z}{\partial x}\,dy\right) + \oint_{PA} \left(\frac{\partial z}{\partial y}\,dx + \frac{\partial z}{\partial x}\,dy\right) = 0$$

$$\Rightarrow \oint_{AB} \left(\frac{\partial z}{\partial y}\,dx + \frac{\partial z}{\partial x}\,dy\right) + \int_B^P \left(-\frac{\partial z}{\partial y}\,dy - \frac{\partial z}{\partial x}\,dx\right) + \int_P^A \left(\frac{\partial z}{\partial y}\,dy + \frac{\partial z}{\partial x}\,dx\right) = 0$$

[$\because$ along BP.. $x + y = \xi + \eta \Rightarrow dx = -dy$
along PA. $x - y = \xi - \eta \Rightarrow dx = dy$]

$$\Rightarrow \oint_{AB} \left(\frac{\partial z}{\partial y}\,dx + \frac{\partial z}{\partial x}\,dy\right) - \int_B^P dz + \int_P^A dz = 0$$

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$$\Rightarrow \oint_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) - (z_P - z_B) + z_A - z_P = 0$$

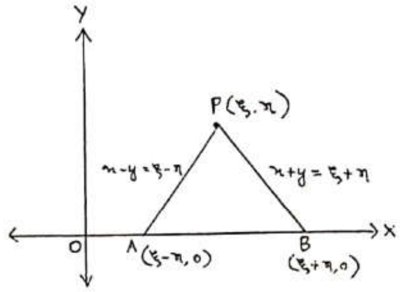
$$\Rightarrow z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\oint_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right)$$

which is the required solution of (i) at any point P.

- Find $z = z(x,y)$ s.t. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$

and $z = f(x)$ ~~and~~, $z_y = g(x)$ on $y = 0$ i.e. x-axis.

$\Rightarrow$



The given equation is .... $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ ---- (i)

$z(x,0) = f(x)$, i.e. $z = f(x)$ on $y = 0$. – (ii)

$\left.\frac{\partial z}{\partial y}\right|_{y=0} = g(x)$, i.e. $z_y = g(x)$ on $y = 0$ -- (iii)

comparing (i) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$ we get..

$R = 1$, $S = 0$, $T = -1$.

Then λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to..

$\lambda^2 - 1 = 0$

$\Rightarrow \lambda = \pm 1$

Then the corresponding characteristic equations are..

$$\left.\begin{aligned} \frac{dy}{dx} + 1 = 0 &\Rightarrow x + y = c_1 \\ \frac{dy}{dx} - 1 = 0 &\Rightarrow x - y = c_2 \end{aligned}\right\} \text{---- (iv)}$$

Let $P(\xi,\eta)$ be any point on xy-plane, we now obtain characteristic of (i) passing through P

so, putting $x = \xi$ and $y = \eta$ in (iv) we get

$c_1 = \xi + \eta$, $c_2 = \xi - \eta$

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The characteristic of (i) passing through P are given by

$$x+y = \xi+\eta$$
$$x-y = \xi-\eta$$

which are shown by the straight lines PB and PA respectively.

Let c' denotes the closed curve ... c': PABP which are made up line PA, the x-axis and line BP.

Let S be the area enclose by c'.

Equation (i) can be written as

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{- - (v)}$$

integrating both sides of (v) over S we get

$$\iint_S \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) dx\,dy = 0$$

$$\Rightarrow \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\,dy = 0$$

$$\Rightarrow \oint_{c'} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) = 0$$

$$\Rightarrow \int_{AB} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) = 0$$

$$\Rightarrow \int_{AB} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_B^P \left(-\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) + \int_P^A \left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx\right) = 0$$

[along BP .. $x+y=\xi+\eta \Rightarrow dx=-dy$
along PA ... $x-y=\xi-\eta \Rightarrow dx = dy$]

$$\Rightarrow \int_{AB} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) - \int_B^P dz + \int_P^A dz = 0$$

$$\Rightarrow \int_c \frac{\partial z}{\partial y}dx - z_P + z_B + z_A - z_P = 0$$

$$\Rightarrow z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_c g(x)\,dx$$

which is the required solution of (i) at any point P.

• Find the solution of one-dimensional non-homogeneous wave equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = f(x,y)$$

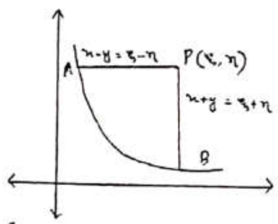
z, z_x, z_y are ~~prescribed~~ described along a given curve C.

$\Rightarrow$ $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = f(x,y)$ -- (i)

comparing (i) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$ we get

$R = 1$, $S = 0$, $T = -1$

Then the corresponding λ-quadratic reduces to



$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

The corresponding characteristic equations are.

$$\left.\begin{aligned} \frac{dy}{dx} + 1 &= 0 \Rightarrow x+y = c_1 \\ \frac{dy}{dx} - 1 &= 0 \Rightarrow x-y = c_2 \end{aligned}\right\} \text{(ii)}$$

Let $P(\xi,\eta)$ be any point on xy-plane, we now obtain characteristic of (i) passing through P

so, putting $x = \xi$ and $y = \eta$ in (ii) we get..

$$c_1 = \xi + \eta,$$
$$c_2 = \xi - \eta$$

$\therefore$ The characteristic of (i) passing through P are given by..

$$x + y = \xi + \eta$$
$$x - y = \xi - \eta$$

which are shown by the straight lines PB and PA respectively.

Let c' ~~be~~ denotes the closed curve : PABP which are made up of line PA, curve C and line BP

Let S be the area enclose by c'.

~~Equation (i) can be written as - -~~

integrating both sides of (i) over S. we get..

$$\iint_S \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) dx\,dy = \iint_S f(x,y)\,dx\,dy$$

$$\iint_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \right\} dx\,dy = \iint_S f(x,y)\,dx\,dy$$

$$\Rightarrow \oint_C \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) = \iint_S f(x,y)\,dx\,dy$$

$$\Rightarrow \int_{AB} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \int_{BP} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \int_{PA} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) = \iint_S f(x,y)\,dx\,dy$$

$$\Rightarrow \int_{AB} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \int_B^P \left(-\frac{\partial z}{\partial y} dy - \frac{\partial z}{\partial x} dx \right) + \int_{AP}^{PA} \left(\frac{\partial z}{\partial y} dy + \frac{\partial z}{\partial x} dx \right) = \iint_S f(x,y)\,dx\,dy$$

[along PA, $x - y = \xi - \eta \Rightarrow dx = dy$
along BP $x + y = \xi + \eta \Rightarrow dx = -dy$]

$$\Rightarrow \int_C \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) - \int_B^P dz + \int_P^A dx = \iint_S f(x,y)\,dx\,dy$$

$$\Rightarrow \int_C \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) - (z_P - z_B) + (z_A - z_P) = \iint_S f(x,y)\,dx\,dy$$

$$\Rightarrow z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_C \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) - \frac{1}{2}\iint_S f(x,y)\,dx\,dy$$

which is the required solution of ~~eq~~ (i) at any point P.

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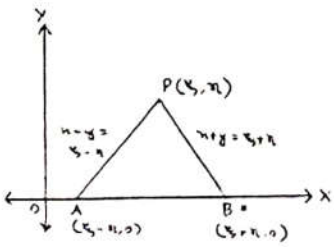
$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 1, \quad \text{when } z(x,0) = \sin x$$
$$z_y(x,0) = x$$

$\Rightarrow$



The given equation is .. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 1$ --- (i)

$z(x,0) = \sin x$ -- (ii)

$\left.\frac{\partial z}{\partial y}\right|_{y=0} = x$ --- (iii)

comparing (i) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$ we get. -

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$R = 1, \; S = 0, \; T = -1$

The λ- quadratic reduces to ....

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

The characteristic equations are given by.

$$\left.\begin{aligned} \frac{dy}{dx} + 1 = 0 &\Rightarrow x + y = c_1 \\ \frac{dy}{dx} - 1 = 0 &\Rightarrow x - y = c_2 \end{aligned}\right\} \text{(iv)}$$

Let $P(\xi, \eta)$ be any point on xy-plane, we now obtain characteristic of (i) passing through P.

so, putting $x = \xi$ and $y = \eta$ in (iv) we get–

$$c_1 = \xi + \eta$$

$$c_2 = \xi - \eta$$

$\therefore$ The characteristic of (i) passing through P are given by--

$$x + y = \xi + \eta$$

$$x - y = \xi - \eta$$

which are shown by the straight lines PB and PA respectively

Let c' denotes the closed curve : PABP which are made up of line PA, x-axis and line BP

Let S be the area enclose by c'

integrating both sides of (i) over S we get..

$$\iint_S \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) dx\,dy = \iint_S dx\,dy$$

$$\Rightarrow \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\,dy = \iint_S dx\,dy$$

$$\Rightarrow \oint_{c'} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) \cancel{dx\,dy} = \iint_S dx\,dy$$

$$\Rightarrow \int_{AB} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) \cancel{dx\,dy} + \int_{BP} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) \cancel{dx\,dy}$$

$$+ \int_{PA} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) = \iint_S dx\,dy$$

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$$\Rightarrow \int_{AB} \frac{\partial z}{\partial y}\,dx + \int_{BP}\left(-\frac{\partial z}{\partial y}\,dy - \frac{\partial z}{\partial x}\,dx\right) + \int_{PA}\left(\frac{\partial z}{\partial y}\,dy + \frac{\partial z}{\partial x}\,dx\right) = \iint_S dx\,dy$$

[∴ along x-axis $y=0 \Rightarrow dy=0$
along BP. $x+y = \xi+\eta \Rightarrow dx = -dy$
along PA. $x-y = \xi-\eta \Rightarrow dx = dy$]

$$\Rightarrow \int_{AB} \frac{\partial z}{\partial y}\,dx - \int_B^P dz + \int_P^A dz = \iint_S dx\,dy$$

$$\Rightarrow \int_{AB} \frac{\partial z}{\partial y}\,dx - z_P + z_B + z_A - z_P = \iint_S dx\,dy$$

$$\Rightarrow z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_{AB} \frac{\partial z}{\partial y}\,dx - \frac{1}{2}\iint_S dx\,dy$$

$$\Rightarrow z_P = \frac{1}{2}\left(\sin(\xi-\eta) + \sin(\xi+\eta)\right) + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} x\,dx - \cancel{\frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} dx \int_0^{\eta} dy}\ \frac{1}{2}\eta^2$$

$$\Rightarrow z_P = \frac{1}{2}\ 2\sin\xi\cos\eta + \frac{1}{4}\left\{(\xi+\eta)^2 - (\xi-\eta)^2\right\} - \frac{1}{2}\cdot\cancel{2\eta}\cdot\eta^2$$

$$\Rightarrow z_P = \sin\xi\cos\eta + \xi\eta - \cancel{\xi^2}\ \frac{1}{2}\eta^2$$

∴ The required solution is ----

$$z(x,y) = \sin x\cos y + xy - \frac{1}{2}y^2$$

• Let $u = \psi(x,t)$ be the solution of the initial value problem

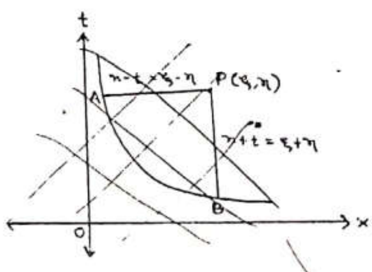
$$u_{tt} = u_{xx} \quad \text{for } -\infty < x < \infty,\ t > 0$$

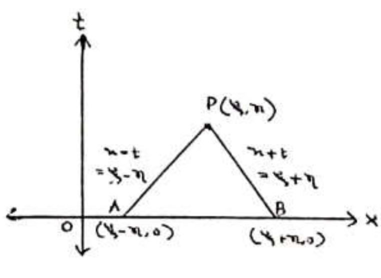
$$u(x,0) = \sin x$$

$$u_t(x,0) = \cos x$$

Find $\psi(\pi/2, \pi/6)$

⇒





The given equation is $\quad \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial t^2} = 0$ --- (i)

$u(x,0) = \sin x$ ---- (ii)

$u_t(x,0) = \cos x$ ---- (iii)

comparing (i) with $Ru_{xx} + Su_{xt} + Tu_{tt} + f(x,t,u,p,q) = 0$ we get

$$R = 1 \quad S = 0 \quad T = -1$$

The λ-quadratic reduces to

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

The characteristic equation are given by...

$$\left.\begin{array}{l} \frac{dt}{dx} + 1 = 0 \Rightarrow x+t = c_1 \\ \\ \frac{dt}{dx} - 1 = 0 \Rightarrow x-t = c_2 \end{array}\right\} \text{--(iv)}$$

Let $P(\xi, \eta)$ be any point on xt-plane. we now obtain the characteristic of (i) passing through P

so putting $x = \xi$ $t = \eta$ in (iv) we get..

$$c_1 = \xi + \eta, \quad c_2 = \xi - \eta$$

$\therefore$ The characteristic of (i) passing through P ~~is~~ are given by..

$$x + t = \xi + \eta$$
$$x - t = \xi - \eta$$

which are shown by the straight lines PB and PA respectively.

Let c' denotes the closed curve PABP. which are made up of line PA, x-axis ~~.......~~ and line BP.

Let S denotes the area enclose by c'

integrating both sides of (i) over S we get..

$$\iint_S \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}\right) dx\,dt = 0$$

$$\Rightarrow \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)\right\} dx\,dt = 0$$

$$\Rightarrow \oint_{c'} \left(\frac{\partial u}{\partial t} dx + \frac{\partial u}{\partial x} dt\right) = 0$$

$$\Rightarrow \int_{AB} \left(\frac{\partial u}{\partial t} dx + \frac{\partial u}{\partial x} dt\right) + \int_{BP} \left(\frac{\partial u}{\partial t} dx + \frac{\partial u}{\partial x} dt\right) + \int_{PA} \left(\frac{\partial u}{\partial t} dx + \frac{\partial u}{\partial x} dt\right) = 0$$

$$\int_{BP}\left(-\frac{\partial u}{\partial t}dt - \frac{\partial u}{\partial x}dx\right) + \int_{PA}\left(\frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx\right) = 0$$

[along AB $t = 0$, $dt = 0$

along BP, $x+t = \xi+\eta \Rightarrow dx = -dt$

along PA, $x-t = \xi-\eta \Rightarrow dx = dt$]

$$\frac{\partial u}{\partial t}dx - \int_B^P du + \int_A^{PA} du = 0$$

$$\frac{\partial u}{\partial t}dx - u_P + u_B + u_A - u_P = 0.$$

$$u_P = \frac{1}{2}(u_A+u_B) + \frac{1}{2}\int_{AB}\frac{\partial u}{\partial t}dx$$

$$\Rightarrow u_P = \frac{1}{2}\{\sin(\xi-\eta)+\sin(\xi+\eta)\} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta}\cos x\, dx$$

$$= \frac{1}{2}\sin(\xi-\eta) + \frac{1}{2}\sin(\xi+\eta) + \frac{1}{2}\sin(\xi+\eta) - \frac{1}{2}\sin(\xi-\eta)$$

$$u_P = \sin(\xi+\eta)$$

The required solution is. $u(x,t) = \sin(x+t)$

$\Rightarrow \psi(x,t) = \sin(x+t)$ [since the solution is unique]

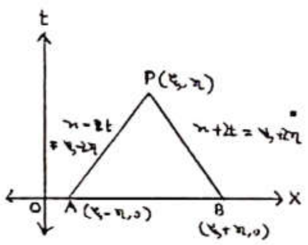
$$\psi(\pi/2, \pi/6) = \sin(\pi/2 + \pi/6)$$

$$= \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}.$$

$u_{tt} = 4u_{xx}$, $-\infty < x < \infty$, $t > 0$

$u(x,0) = x$

$u_t(x,0) = 0$



The given equation.. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{4}\dfrac{\partial^2 u}{\partial t^2} = 0$ -- (1)

$u(x,0) = x$

$u_t(x,0) = 0$

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comparing (i) with $Rr + Ss + Tt + f(x,t,u,p,q) = 0$ we get..

$$R = 1, \quad S = 0 \quad T = -\frac{1}{4}$$

$\therefore$ The λ- quadratic reduces to..

$$\lambda^2 - \frac{1}{4} = 0 \Rightarrow \lambda = \pm\frac{1}{2}$$

$\therefore$ The characteristic equation are given by..

$$\left.\begin{aligned} \frac{dt}{dx} + \frac{1}{2} &\Rightarrow x + 2t = c_1 \\ \frac{dt}{dx} - \frac{1}{2} &\Rightarrow x - 2t = c_2 \end{aligned}\right\} \text{(ii)}$$

Let $P(\xi, \eta)$ be any point on xt-plane, we now obtain the characteristic of (i) passing through P

so, putting $x = \xi,\ t = \eta$ in (ii), we get.-

$$c_1 = \xi + 2\eta$$

$$c_2 = \xi - 2\eta$$

The characteristic of (i) passing through P are given by..

$$x + 2t = \xi + 2\eta$$

$$x - 2t = \xi - 2\eta$$

which are shown by the lines PB and PA respectively.

Let c' denotes the closed curve PABP which are made up line PA, x-axis and line BP

Let S be the area enclose by c'

integrating both sides of (1) over S, we get..

$$\iint_S \left(\frac{\partial^2 u}{\partial x^2} - \frac{1}{4}\frac{\partial^2 u}{\partial t^2}\right) dx\, dt = 0$$

$$\Rightarrow \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial t}\left(\frac{1}{4}\frac{\partial u}{\partial t}\right)\right\} dx dt = 0$$

$$\Rightarrow \oint_{c'} \left(\frac{1}{4}\frac{\partial u}{\partial t} dx + \frac{\partial u}{\partial x} dt\right) = 0$$

$$\Rightarrow \int_{AB} \left(\frac{1}{4}\frac{\partial u}{\partial t} dx + \frac{\partial u}{\partial x} dt\right) + \int_{BP} \left(\frac{1}{4}\frac{\partial u}{\partial t} dx + \frac{\partial u}{\partial x} dt\right)$$

$$+ \int_{PA} \left(\frac{1}{4}\frac{\partial u}{\partial t} dx + \frac{\partial u}{\partial x} dt\right) = 0$$

$$= \int \frac{1}{4}\frac{\partial u}{\partial t}\,dx + \int_{BP}\left(\frac{1}{4}\frac{\partial u}{\partial t}\,dx + \frac{1}{4}\frac{\partial u}{\partial x}\,dt\right) + \int_{BP}\frac{3}{4}\frac{\partial u}{\partial x}\,dt + \int_{PA}\left(\frac{1}{4}\frac{\partial u}{\partial t}\,dx + \frac{1}{4}\frac{\partial u}{\partial x}\,dt\right) + \int_{PA}\frac{3}{4}\frac{\partial u}{\partial x}\,dt = 0$$

[along AB ~~y=0~~ t = 0, dt = 0

$$\int_{AB}\frac{1}{4}\frac{\partial u}{\partial t}\,dx + \int_{BP}\left(-\frac{1}{4}\right)\left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\right) - \frac{3}{4}\int_{BP}\frac{\partial u}{\partial x}\,dx + \int_{PA}\frac{1}{4}\left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\right) + \frac{3}{4}\int_{PA}\frac{\partial u}{\partial x}\,dx = 0$$

[~~along BP x+2t~~

$$= \int_{AB}\frac{1}{4}\frac{\partial u}{\partial t}\,dx + \int_{BP}\left(\frac{1}{4}\frac{\partial u}{\partial t}(-2\,dt) + \frac{\partial u}{\partial x}\left(-\frac{dx}{2}\right)\right) + \int_{PA}\left(\frac{1}{4}\frac{\partial u}{\partial t}\cdot 2\,dt + \frac{\partial u}{\partial x}\cdot\frac{1}{2}\,dx\right) = 0$$

[$\because$ along AB $t=0$. $dt=0$

along BP, $x+2t = \xi + 2\eta \Rightarrow dx = -2dt$

along PA, $x-2t = \xi - 2\eta \Rightarrow dx = 2dt$]

$$= \int_{AB}\frac{1}{4}\frac{\partial u}{\partial t}\,dx - \frac{1}{2}\int_{B}^{P} du + \frac{1}{2}\int_{P}^{A} du = 0$$

$$\int_{AB}\frac{1}{4}\frac{\partial u}{\partial t}\,dx - \frac{1}{2}u_P + \frac{1}{2}u_B + \frac{1}{2}u_A - \frac{1}{2}u_P = 0$$

$$\Rightarrow u_P = \frac{1}{2}(u_A + u_B) + \frac{1}{4}\int_{AB}\frac{\partial u}{\partial t}\,dx$$

$$\Rightarrow u_P = \frac{1}{2}(u_A + u_B) \quad \left[\because \left.\frac{\partial u}{\partial t}\right|_{(x,0)} = 0\right].$$

$$\Rightarrow u_P = \frac{1}{2}(\xi - \eta + \xi + \eta)$$

$$\Rightarrow u_P = \xi$$

The required solution is ... $u(x,t) = x$

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- Solution of PDE using laplace transformation :-

$$L\{u(x,t)\} = U(x,s) = \int_0^\infty e^{-st}\, u(x,t)\, dt$$

$$L\left\{\frac{\partial u}{\partial t}, s\right\} = \int_0^\infty e^{-st}\, \frac{\partial u}{\partial t}\, dt$$

$$= \lim_{p\to\infty} \int_0^p e^{-st}\, \frac{\partial u}{\partial t}\, dt$$

$$= \lim_{p\to\infty} \left\{\left[e^{-st}\, u(x,t)\right]_0^p + s\int_0^p e^{-st}\, u(x,t)\, dt\right.$$

$$= s \int_0^\infty e^{-st}\, u(x,t)\, dt - u(x,0)$$

$$= s\, U(x,s) - u(x,0)$$

$$L\left\{\frac{\partial u}{\partial x}\right\} = \int_0^\infty e^{-st}\, \frac{\partial u}{\partial x}\, dt$$

$$= \frac{\partial}{\partial x} \int_0^\infty e^{-st}\, u(x,t)\, dt$$

$$= \frac{d}{dx} U(x,s)$$

$$L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2}{dx^2} U(x,s)$$

$$L\left\{\frac{\partial^2 u}{\partial t^2} : s\right\}$$

$$= L\left\{\frac{\partial V}{\partial t} : s\right\}, \quad V = \frac{\partial u}{\partial t}$$

$$= s\, L\{V : s\} - V(x,0)$$

$$= s\left(s\, U(x,s) - u(x,0)\right) - u_t(x,0)$$

$$= s^2 U(x,s) - s\, u(x,0) - u_t(x,0)$$

$$L\left[\frac{\partial^2 u}{\partial x \partial t}\right] = L\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)\right]$$

$$= \frac{d}{dx}\left\{s\, U(x,s) - u(x,0)\right\}$$

$$= s\frac{dU}{dx} - \frac{du}{dx}(x,0)$$

- Solve by using laplace transformation method..

PDE: $u_t = u_{xx}$, $0<x<1$, $t>0$

BCs : $u(0,t) = 1$,

$u(1,t) = 1$, $t>0$

IC : $u(x,0) = 1 + \sin \pi x$, $0<x<1$

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The given PDE is. $u_t = u_{xx}$ - - (i)

Taking the laplace transformation of both sides of (i) we get

$$\cancel{s\frac{du}{dx} - \frac{du}{}u(x,0) = \frac{d^2u}{dx^2}}$$

$$\Rightarrow \cancel{\frac{d^2u}{dx^2} - s\frac{du}{}} \qquad sU(x,s) - u(x,0) = \frac{d^2U}{dx^2}$$

$$\Rightarrow \frac{d^2U}{dx^2} - sU(x,s) = -(1+\sin\pi x)$$

$\therefore$ CF is... $Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}$

$$PI = \frac{1}{D^2-s}\cdot -(1+\sin\pi x) \quad \text{where } D \equiv \frac{d}{dx}$$

$$= -\frac{1}{D^2-s}1 \cdot - \frac{1}{D^2-s}\sin\pi x$$

$$= \frac{1}{s} + \frac{\sin\pi x}{\pi^2+s}$$

$\therefore$ The GS is... $U(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x} + \frac{1}{s} + \frac{\sin\pi x}{\pi^2+s}$ --- (ii)

Now from BC.s we get... $u(0,t) = 1$ and $u(1,t) = 1$.

Taking their laplace transformation we get...

$$U(0,s) = \frac{1}{s} \quad \text{and} \quad U(1,s) = \frac{1}{s}$$

Now, we have... $A + B = 0$

$$Ae^{\sqrt{s}} + Be^{\sqrt{s}} = 0$$

This is a homogeneous system the determinant of the co-efficient matrix is..

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{s}} & e^{-\sqrt{s}} \end{vmatrix} = e^{-\sqrt{s}} - e^{\sqrt{s}} \neq 0$$

Thus, the only possible solution is the trivial & is....

$A = B = 0$

$$U(x,s) = \frac{1}{s} + \frac{\sin \pi x}{\pi^2 + s}$$

Taking the inverse laplace transformation we get.

$$u(x,t) = L^{-1}\left(\frac{1}{s}, t\right) + L^{-1}\left[\frac{\sin \pi x}{\pi^2+s}, t\right]$$

$$= 1 + \sin \pi x \, e^{-\pi^2 t}$$

∴ The required general solution is ---

$$u(x,t) = 1 + \sin \pi x \, e^{-\pi^2 t}$$

- $u_{tt} = u_{xx}$, $0 < x < 1$, $t > 0$

$u(0,t) = u(1,t) = 0$, $t > 0$

$u(x,0) = \sin \pi x$, $u_t(x,0) = -\sin \pi x$, $0 < x < 1$

⇒ The given PDE is --- $u_{tt} = u_{xx}$ -- (i)

Taking the laplace transformation of both sides of (i), we get...

$$\frac{d^2 U}{dx^2} = s^2 U(x,s) - s\,u(x,0) - u_t(x,0)$$

$$\Rightarrow \frac{d^2 U}{dx^2} - s^2 U = (1-s)\sin \pi x$$

∴ C.F. is .. $Ae^{sx} + Be^{-sx}$

$$\text{P.I.} = \frac{1}{D^2 - s^2}(1-s)\sin \pi x \quad \text{where } D \equiv \frac{d}{dx}$$

$$= \frac{s-1}{\pi^2 + s^2}\sin \pi x$$

∴ G.S is .. $U(x,s) = Ae^{sx} + Be^{-sx} + \frac{s-1}{x^2+s^2}\sin \pi x$ -- (ii)

Now from BC, we get.. $u(0,t) = u(1,t) = 0$

Taking their laplace transformation we get.

$$U(0,s) = U(1,s) = 0$$

∴ We have --

$$A + B = 0$$
$$Ae^{s} + Be^{-s} = 0 \quad \Rightarrow A = B = 0$$

$$u(x,s) = \frac{s-1}{\pi^2+s^2} \sin \pi x$$

Taking the ⅰ inverse laplace transformation we get.

$$u(x,t) = \sin \pi x \left[L^{-1}\left\{\frac{s}{\pi^2+s^2}\right\} - L^{-1}\left\{\frac{1}{\pi^2+s^2}\right\}\right]$$

$$= \sin \pi x \left(\cos \pi t - \frac{\sin \pi t}{\pi}\right)$$

∴ The required general solution is ...

$$u(x,t) = \sin \pi x \left(\cos \pi t - \frac{\sin \pi t}{\pi}\right)$$

• $u_{xx} - 2u_{xy} + u_{yy} + u_x - u_y = e^{x+y}$

$\Rightarrow$ $u_{xx} - 2u_{xy} + u_{yy} + u_x - u_y = e^{x+y}$ . . (1)

comparing it with $R\,u_{xx} + S\,u_{xy} + T\,u_{yy} + g(x, y, u, u_x, u_y) = 0$ we get .

$$R = 1, \quad S = -2, \quad T = 1$$

$S^2 - 4RT = 4 - 4 = 0$ shows that (1) is parabolic

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to . .

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0 \Rightarrow \lambda = 1, 1$$

The corresponding characteristic equation is . .

$$\frac{dy}{dx} + 1 = 0 \Rightarrow y + x = c_1$$

We choose ξ, η s.t. $\xi = y + x$, $\eta = y - x$ s.t $\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$= u_\xi - u_\eta$$

$$u_{xx} = \frac{\partial}{\partial x}(u_\xi - u_\eta)$$

$$= \frac{\partial}{\partial \xi}(u_\xi - u_\eta) \cdot \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(u_\xi - u_\eta) \cdot \frac{\partial \eta}{\partial x}$$

$$= u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$= u_\xi + u_\eta$$

$$u_{yy} = \frac{\partial}{\partial y}(u_\xi + u_\eta)$$

$$= \frac{\partial}{\partial \xi}(u_\xi + u_\eta)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}(u_\xi + u_\eta)\frac{\partial \eta}{\partial y}$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = \frac{\partial}{\partial x}(u_\xi + u_\eta)$$

$$= \frac{\partial}{\partial \xi}(u_\xi + u_\eta)\cdot\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(u_\xi + u_\eta)\frac{\partial \eta}{\partial x}$$

$$= u_{\xi\xi} - u_{\eta\eta}$$

putting this in (i) we get . .

$$u_{\cancel{\xi\xi}} - 2u_{\cancel{\xi}\eta} + \overset{u_{\eta\eta}}{\cancel{u_{\eta\eta}}} - 2u_{\cancel{\xi\xi}} + 2u_{\eta\eta} + u_{\cancel{\xi\xi}} + 2u_{\cancel{\xi}\eta} + u_{\eta\eta} + u_{\cancel{\xi}} - u_\eta$$

$$- u_{\cancel{\xi}} - u_\eta = e^{\xi}$$

$$\Rightarrow \quad 4u_{\eta\eta} - 2u_\eta = e^{\xi}$$

$\Rightarrow$ which is the required cannonical form.

putting $u_\eta = z$. then

$$\frac{\partial z}{\partial \eta} - \frac{1}{2}z = e^{\xi} \quad \Rightarrow \quad \frac{dz}{d\eta} - \frac{1}{2}z = e^{\xi}$$

which is linear in z.

$$\therefore \text{ I.F. } = e^{\int -\frac{1}{2}d\eta} = e^{-\frac{1}{2}\eta}$$

$$\therefore \cancel{e^{-\frac{1}{2}\eta}dz - e^{-\frac{1}{2}\eta}\cdot\frac{1}{2}z\,d\eta = e^{\xi}}\,d\eta$$

$$\Rightarrow \cancel{d(\eta e^{-\frac{1}{2}\eta}) = e^{\xi}\eta + F(\xi)}$$

$$e^{-\frac{1}{2}\eta}dz - \frac{1}{2}z\,e^{-\frac{1}{2}\eta}d\eta = e^{\xi}\cdot e^{-\frac{1}{2}\eta}d\eta$$

$$\Rightarrow \int d(z e^{-\frac{1}{2}\eta}) = \int e^{\xi}\cdot e^{-\frac{1}{2}\eta}d\eta$$

$$z e^{-\frac{1}{2}\eta} = e^{\xi}\cdot\cancel{\xi}(-2)e^{-\frac{1}{2}\eta} + F(\xi)$$

$$z = F(\xi)e^{\frac{1}{2}\eta} - 2e^{\xi}$$

$$\frac{\partial u}{\partial \eta} = F(\xi)e^{\frac{1}{2}\eta} - 2e^{\xi}$$

$$\Rightarrow u = F(\xi)\,2e^{\frac{1}{2}\eta} - 2\eta e^{\xi} + G(\xi)$$

$$\Rightarrow u = 2f(y+x)\cdot e^{(y-x)/2} - 2(y-x)e^{y+x} + g(y+x)$$

- $4U_{xx} + 5U_{xy} + U_{yy} + U_x + U_y = 2$

$\Rightarrow$ $4U_{xx} + 5U_{xy} + U_{yy} + U_x + U_y = 2 \quad \ldots (i)$

comparing (i) with $RU_{xx} + SU_{xy} + TU_{yy} + g(x, y, u, u_x, u_y) = 0$ we get

$$R = 4, \quad S = 5 \quad T = 1$$

$S^2 - 4RT = 25 - 16 = 9 > 0$ shows that (i) is hyperbolic

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to.

$$4\lambda^2 + 5\lambda + 1 = 0$$

$$\Rightarrow 4\lambda^2 + 4\lambda + \lambda + 1 = 0$$

$$\Rightarrow 4\lambda(\lambda+1) + (\lambda+1) = 0$$

$$\Rightarrow (\lambda+1)(4\lambda+1) = 0$$

$$\Rightarrow \lambda = -1, -\tfrac{1}{4}$$

$\therefore$ The corresponding characteristic equations are

$$\frac{dy}{dx} - 1 = 0 \Rightarrow y - x = c_1$$

$$\frac{dy}{dx} - \frac{1}{4} = 0 \Rightarrow 4y - x = c_2$$

We choose ξ, η s.t. $y - x = \xi$; $4y - x = \eta$ s.t. $\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0$

$\therefore$ ~~U_ξ~~ $U_x =$ $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$

$$= -U_\xi - U_\eta$$

$$U_{xx} = \frac{\partial}{\partial x}(-U_\xi - U_\eta)$$

$$= \frac{\partial}{\partial \xi}(-U_\xi - U_\eta)\cdot\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(-U_\xi - U_\eta)\cdot\frac{\partial \eta}{\partial x}$$

$$= U_{\xi\xi} + U_{\xi\eta} + U_{\xi\eta} + U_{\eta\eta}$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

$$U_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= U_\xi + 4U_\eta$$

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$$u_{yy} = \frac{\partial}{\partial y}\left(u_\xi + 4u_\eta\right)$$

$$= \frac{\partial}{\partial \xi}\left(u_\xi + 4u_\eta\right)\cdot\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(u_\xi + 4u_\eta\right)\frac{\partial \eta}{\partial y}$$

$$= u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\xi\eta} + 16u_{\eta\eta}$$

$$= u_{\xi\xi} + 8u_{\xi\eta} + 16u_{\eta\eta}$$

$$\therefore\ u_{xy} = \frac{\partial}{\partial y}\left(-u_\xi - u_\eta\right)$$

$$= \frac{\partial}{\partial \xi}\left(-u_\xi - u_\eta\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(-u_\xi - u_\eta\right)\cdot\frac{\partial \eta}{\partial y}$$

$$= -u_{\xi\xi} - u_{\xi\eta} - 4u_{\xi\eta} - 4u_{\eta\eta}$$

$$= -u_{\xi\xi} - 5u_{\xi\eta} - 4u_{\eta\eta}$$

putting this value in (i) we get . .

$$4\cancel{u_{\xi\xi}} + 8u_{\xi\eta} + 4\cancel{u_{\eta\eta}} + -5\cancel{u_{\xi\xi}} - 25u_{\xi\eta} - 20\cancel{u_{\eta\eta}} + \cancel{u_{\xi\xi}} + 8u_{\xi\eta} + 16\cancel{u_{\eta\eta}}$$
$$- \cancel{u_\xi} - u_\eta + \cancel{u_\xi} + 4u_\eta = 2$$

$$\Rightarrow\ -9u_{\xi\eta} + 3u_\eta = 2$$

$$\Rightarrow\ u_{\xi\eta} - \frac{1}{3}u_\eta = -\frac{2}{9}$$

which is the required cannonical form.

Now put $u_\eta = z$, then -

$$\frac{dz}{\partial \xi} - \frac{1}{3}z = -\frac{2}{9}$$

which is linear in z.

$$\therefore\ \text{T.F} = e^{-\int \frac{1}{3} d\xi} = e^{-\frac{1}{3}\xi}$$

$$\therefore\ e^{-\frac{1}{3}\xi}dz - \frac{1}{3}e^{-\frac{1}{3}\xi}z\,d\xi = -\frac{2}{9}e^{-\frac{1}{3}\xi}d\xi$$

$$\Rightarrow \int d\left(z e^{-\frac{1}{3}\xi}\right) = \int -\frac{2}{9}e^{-\frac{1}{3}\xi}d\xi$$

$$z e^{-\frac{1}{3}\xi} = \frac{2}{3}e^{-\frac{1}{3}\xi} + F(\eta)$$

$$\therefore \quad z = \frac{2}{3} + F(\eta)\, e^{-\frac{1}{3}\xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{2}{3} + F(\eta)\, e^{-\frac{1}{3}\xi}$$

$$u = \frac{2}{3}\eta + e^{-\frac{1}{3}\xi} \cancel{\int F(\eta)}\; G(\eta) + H(\xi)$$

$$\Rightarrow \cancel{u = \frac{2}{3}(y-x)} + e^{-}$$

$$\Rightarrow \quad u = \frac{2}{3}(4y-x) + e^{-\frac{1}{3}(y-x)}\, F(4y-x) + g(y-x)$$

1+1+1

1+1+1   1+1

12   2

1+1   60