using the boundary condition .. $T = 20$ when $x = 0$, we get.

$$20 = 20 + Be^{-\alpha\lambda^2 t}$$

$$\Rightarrow B = 0$$

Again using the boundary condition $T = 60$ when $x = 10$, we get.

$$60 = 60 + e^{-\alpha\lambda^2 t}\, C \sin \lambda x 10$$

$$\Rightarrow \sin \lambda 10 = 0 = \sin n\pi$$

$$\Rightarrow \lambda . 10 = n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{10}, \quad n = 1, 2.$$

$\therefore$ The complete solution is given by.

$$T(x,t) = 4x + 20 + \sum_{n=1}^{\infty} C_n \exp\left(-\alpha \frac{n^2\pi^2}{100} t\right) \sin \frac{n\pi x}{10}$$

Now using the initial condition $T = 10x$, when $t = 0$, we obtain.

$$10x = 4x + 20 \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{10}$$

$$\Rightarrow 6x - 20 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{10}$$

where $\quad C_n = \frac{2}{10} \int_0^{10} (6x - 20) \sin \frac{n\pi x}{10}\, dx$

$$= \frac{1}{5} \int_0^{10} \left(6x \sin \frac{n\pi x}{10} - 20 \sin \frac{n\pi x}{10}\right) dx$$

$$= \frac{1}{5}\left[6\left\{\left(-x \frac{\cos n\pi x/10}{n\pi/10}\right)\Bigg|_0^{10} + \int_0^{10} \frac{\cos \frac{n\pi x}{10}}{n\pi/10}\, dx\right\}\right] - \frac{1}{5}\left[20\left(-\frac{\cos n\pi x/10}{n\pi/10}\right)\right]_0^{10}$$

$$= \frac{1}{5}\left[6\left(-\frac{x \cos n\pi x/10}{n\pi/10} + \frac{\sin n\pi x/10}{(n\pi/10)^2}\right]_0^{10} + 4\left[\frac{\cos n\pi x/10}{n\pi/10}\right]_0^{10}$$

$$= \frac{1}{5}\left(-\frac{60}{n\pi/10}(-1)^n\right) + 4\frac{10}{n\pi}\left\{(-1)^n - 1\right\}$$

$$= -\frac{1}{5}\left(\frac{600}{n\pi}(-1)^n - \frac{200}{n\pi}(-1)^n + \frac{200}{n\pi}\right)$$

$$= -\frac{1}{5}\left[(-1)^n \frac{400}{n\pi} + \frac{200}{n\pi}\right]$$

The required solution is..

$$T(x,t) = 4x + 20 - \frac{1}{5}\sum_{n=1}^{\infty}\left[(-1)^n \frac{400}{n\pi} + \frac{200}{n\pi}\right] \exp\left(-\alpha \frac{n^2\pi^2 t}{100}\right) \sin\frac{n\pi x}{10}$$

The boundaries of a rectangle $0 \le x \le a$, $0 \le y \le b$ are maintained at 0 temperature. At $t=0$, the term T has the prescribed value $f(x,y)$. Show that for $t>0$ the temperature at a point within the rectangle is given by...

$$T(x,y,t) = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) \exp(-\alpha \lambda_{mn}^2 t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

⇒ The problem is to solve the diffusion equation described by..

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \quad 0<x<a, \ 0<y<b, \ t>0 \quad \ldots (i)$$

BCs are.... $T(0,y,t) = T(a,y,t) = 0 \quad 0<y<b, \ t>0$

$T(x,0,t) = T(x,b,t) = 0 \quad 0<x<a, \ t>0$

Initial condition is.... $T(x,y,0) = f(x,y)$, $0<x<a$, $0<y<b$

Let $T(x,y,t) = X(x).Y(y).B(t)$

$\cancel{X''YB}$

$(X''YB + XY''B)\alpha = XYB'$

X''

from (i) we get..

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{\alpha}\frac{B'}{B} = -\lambda^2$$

$\therefore B' + \alpha\lambda^2 B = 0$

$\Rightarrow B = a e^{-\alpha\lambda^2 t}$

Again, we get.

$$\frac{X''}{X} = -\left(\lambda^2 + \frac{Y''}{Y}\right) = -p^2 \text{ (say)}$$

$\therefore X'' + p^2 X = 0 \Rightarrow A\cos px + B\sin px = X$

$Y'' + q^2 Y = 0 \Rightarrow C\cos qy + D\sin qy = Y$ where $q^2 = \lambda^2 - p^2$

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Now using BC. $T(0,y,t) = 0 = T(a,y,t)$ we get.

$$A = 0$$

$$B \sin pa = 0 \qquad [\because \text{ for non-trivial solution } B \neq 0]$$

$$\Rightarrow \sin pa = 0 = \sin m\pi$$

$$\Rightarrow pa = m\pi$$

$$\Rightarrow p = \frac{m\pi}{a}$$

$$\therefore T(x,y,T) = B \sin \frac{m\pi x}{a} (C \cos qy + D \sin qy) e^{-\alpha\lambda^2 t}$$

Now, using BC. $T(x,0,t) = T(x,b,t)$ we get..

$$C = 0$$

$$D \sin qb = 0$$

$$\Rightarrow \sin qb = 0 = \sin n\pi$$

$$\Rightarrow qb = n\pi$$

$$\Rightarrow q = \frac{n\pi}{b}$$

$$\therefore T(x,y,T) = BD \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\alpha\lambda^2 t}$$

$m, n = 1, 2 \ldots$

$\therefore$ The complete solution is given by..

$$T(x,y,T) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \exp(-\alpha \lambda_{mn}^2 t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

~~where~~ $\cancel{a_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b F(x,y) \sin \frac{m\pi x}{a} \sin}$

Now. $F(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$

where $a_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b F(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx\, dy$

$\therefore$ Hence the required ~~genera~~ general solution is given by.

$$T(x,y,t) = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)\exp(-\alpha\lambda_{mn}^2 t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\text{where } f(m,n) = \int_0^a\int_0^b F(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$\text{and } \lambda_{mn}^2 = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

Reduce it to a cannonical form and solve it :-

$x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} + xy\,u_x + y^2u_y = 0$

$\Rightarrow x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} + xy\,u_x + y^2u_y = 0 \quad \cdots (i)$

comparing (i) with $R\,u_{xx} + S\,u_{xy} + T\,u_{yy} + g(x,y,u,u_x,u_y) = 0$ we get..

$$R = x^2,\quad S = 2xy,\quad T = y^2$$

$\therefore S^2 - 4RT = 4x^2y - 4x^2y^2 = 0$, showing that (i) is parabolic.

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to..

$$x^2\lambda^2 + 2xy\lambda + y^2 = 0$$

$$\Rightarrow (x\lambda + y)^2 = 0$$

$$\Rightarrow \lambda = -y/x,\ -y/x$$

The corresponding characteristic equation is..

$$\frac{dy}{dx} - y/x = 0$$

$$\Rightarrow \log(y/x) = \log c$$

$$\Rightarrow y/x = c$$

Choose $\xi = y/x$ and $\eta = y$ s.t $\dfrac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$= -\frac{y}{x^2}u_\xi$$

$$u_{xx} = \frac{y^2}{x^4}u_{\xi\xi} + \frac{2y}{x^3}u_\xi$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

$$= \frac{1}{x} u_\xi + u_\eta$$

$$\therefore\ u_{yy} = \frac{1}{x^2} u_{\xi\xi} + \frac{2}{x} u_{\xi\eta} + u_{\eta\eta}$$

$$\therefore\ u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{x} u_\xi + u_\eta\right)$$

$$= -\frac{1}{x^2} u_\xi + \frac{1}{x}\frac{\partial}{\partial x}(u_\xi) + \frac{\partial}{\partial x}(u_\eta)$$

$$= -\frac{1}{x^2} u_\xi + \frac{1}{x}\left[\frac{\partial}{\partial \xi}(u_\xi)\cdot\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(u_\xi)\cdot\frac{\partial \eta}{\partial x}\right]$$

$$+ \left[\frac{\partial}{\partial \xi}(u_\eta)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(u_\eta)\cdot\frac{\partial \eta}{\partial x}\right]$$

$$= -\frac{1}{x^2} u_\xi - \frac{y}{x^3} u_{\xi\xi} - \frac{y}{x^2} u_{\xi\eta}$$

putting these in (i) we get ...

$$u_{\eta\eta} + u_\eta = 0$$

which is the required cannonical form.

Now. put $u_\eta = \omega$

$$\frac{\partial \omega}{\partial \eta} + \omega = 0$$

$$\Rightarrow \omega = f(\xi)e^{-\eta}$$

$$\Rightarrow \frac{\partial u}{\partial \eta} = f(\xi)e^{-\eta}$$

$$\Rightarrow u = -f(\xi)e^{-\eta} + g(\xi)$$

$$\Rightarrow u = g(y/x) - e^{-y} f(y/x)$$

which is the required general solution

Solution of Diffusion equation in cylindrical co-ordinates :-

Consider a three - dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

In cylindrical co-ordinates (r, θ, z), it becomes ....

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \quad \text{--- (i)}$$

where $T = T(r, \theta, z, t)$

Let us assume separation of variables in the form....

$$T(r,\theta,z,t) = R(r).H(\theta).Z(z)\,\beta(t)$$

Substituting this in (i) we get.

$$R''HZ\beta + \frac{1}{r}R'HZ\beta + \frac{1}{r^2}H''RZ\beta + Z''RH\beta = \frac{\beta'}{\alpha}RHZ$$

$$\Rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \frac{Z''}{Z} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2 \text{ (say)}$$

$$\beta' + \alpha\lambda^2\beta = 0 \Rightarrow \beta = e^{-\alpha\lambda^2 t}$$

and $$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 = -\frac{Z''}{Z} = -\mu^2 \text{ (say)}$$

$$\therefore Z'' - \mu^2 Z = 0 \Rightarrow \cancel{Ae^{\mu z}} \; Z = Ae^{\mu z} + Be^{-\mu z}$$

and $$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 + \mu^2 = 0$$

$$\Rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} + (\lambda^2+\mu^2)r^2 = -\frac{H''}{H} = \nu^2 \text{ (say)}$$

$$\therefore H'' + \nu^2 H = 0 \Rightarrow H = C\cos\nu\theta + D\sin\nu\theta$$

and $$R'' + \frac{1}{r}R' + \left[(\lambda^2+\mu^2) - \frac{\nu^2}{r^2}\right]R = 0$$

which is a Bessel's equation of order ν and its general solution is.

$$R(r) = c_1 J_\nu(\sqrt{\lambda^2+\mu^2}\,r) + c_2 Y_\nu(\sqrt{\lambda^2+\mu^2}\,r)$$

where $J_\nu(r)$ and $Y_\nu(r)$ are Bessel's functions of order ν of first kind and 2nd kind respectively.

The equation has only one bounded solution is..

$$R(r) = J_\nu(\sqrt{\lambda^2+\mu^2}\, r)$$

∴ The general solution of the equation (i) is given by...

$$T(r,\theta,z,t) = e^{-\alpha\lambda^2 t}\left[Ae^{\mu z}+Be^{-\mu z}\right]\left[C\cos\nu\theta + D\sin\nu\theta\right] J_\nu(\sqrt{\lambda^2+\mu^2}\, r)$$

- $\dfrac{\partial\theta}{\partial t} = \dfrac{\partial^2\theta}{\partial x^2}$ $0 \le x \le a,\ t>0$

subject to the condition...

$$\theta(0,t) = \theta(a,t) = 0$$

$$\theta(x,0) = \theta_0 \text{ (constant)}$$

⇒ The equation is.. $\dfrac{\partial\theta}{\partial t} = \dfrac{\partial^2\theta}{\partial x^2}$ --- (i)

Let the solution of (i) is...

$$\theta(x,t) = X(x).\,T(t)$$

Then the given equation (i) gives....

$$\frac{X''}{X} = \frac{T'}{T} = -\alpha^2 \text{ (say)}$$

$$\therefore\ X'' + \alpha^2 X = 0 \quad \text{--- (ii)}$$

$$T' + \alpha^2 T = 0 \quad \text{--- (iii)}$$

solving (ii) we get..

$$X(x) = A\cos\alpha x + B\sin\alpha x$$

Now condition-1 gives...

$$X(0) = 0 = X(a)$$

using this we get.....

$$A = 0$$

$$B\sin\alpha a = 0$$

If $B = 0$ then the solution is trivial. so for a non-trivial solution $B \neq 0$

$\sin \alpha a = 0 = \sin n\pi$

$\Rightarrow \alpha a = n\pi$

$\Rightarrow \alpha = \frac{n\pi}{a}$

$\alpha_n = \frac{n\pi}{a}$, $n = 1.2. -$

$X(x) = B_n \sin \frac{n\pi x}{a}$, $n = 1.2.3.-$

Now solving (ii) we get.

$$T(t) = c_0 e^{-\alpha^2 t}$$

$$= c_n e^{-\frac{n^2\pi^2}{a^2}t}.$$

The solution is in the form

$$\theta(x,t) = B_n \sin \frac{n\pi x}{a} \, c_n e^{-\frac{n^2\pi^2}{a^2}t}$$

$$= a_n \sin \frac{n\pi x}{a} e^{-\frac{n^2\pi^2}{a^2}t}$$

By the principle of superposition..

$$\theta(x,t) = \sum_{n=1}^{\alpha} a_n \sin \frac{n\pi x}{a} e^{-\frac{n^2\pi^2}{a^2}t}$$

Now from condition-(ii) we get.

$$\theta(x,0) = \theta_0$$

$$\therefore \theta_0 = \sum_{n=1}^{\alpha} a_n \sin \frac{n\pi x}{a}$$

$$\therefore a_n = \frac{2}{a}\int_0^a \theta_0 \sin \frac{n\pi x}{a} dx$$

$$= \frac{2}{a}\theta_0 \left[-\frac{\cos \frac{n\pi x}{a}}{n\pi/a}\right]_0^a$$

$$= \frac{2}{n\pi}\theta_0 (1-(-1)^n)$$

$a_n = 0$ if n is even i.e if $n = 2m$

$= \frac{4}{n\pi}\theta_0$ if n is odd i.e. if $n = 2m-1$

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$\therefore$ The solution is given by..

$$\theta(r,t) = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi} \theta_0 \sin\frac{(2m-1)\pi r}{a} \exp\left(-\frac{(2m-1)^2\pi^2}{a^2} t\right)$$

- $$\frac{\partial\theta}{\partial t} = \nu\left(\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r}\right)$$

subject to...

$r=0$, θ is finite, $t>0$

$r=a$, $\theta=0$, $t>0$

$\theta = \frac{P}{4\mu}(a^2-r^2)$ $t=0$

Here P, μ and ν are constants

$\Rightarrow$ Let $\theta(r,t) = R(r)\,T(t)$

substituting this in the given equation we get..

$$RT' = \nu\left(R''T + \frac{1}{r}R'T\right)$$

$$\therefore \frac{1}{\nu}\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\alpha^2 \text{ (say)}$$

$\therefore$ $T' = -\alpha^2\nu T \Rightarrow T = c_0 e^{-\nu\alpha^2 t}$

and $R'' + \frac{1}{r}R' + \alpha^2 R = 0$

which is a Bessel's equation of order zero, whose solution is.

$$R = A_1 J_0(\alpha r) + B_1 Y_0(\alpha r)$$

Thus the general solution is..

$$\theta(r,t) = [A J_0(\alpha r) + B Y_0(\alpha r)]\exp(-\nu\alpha^2 t)$$

In view of first boundary condition, the solution is..

$$\theta(r,t) = A J_0(\alpha r)\exp(-\nu\alpha^2 t)$$

The 2nd B.C. gives.

$$J_0(\alpha a) = 0$$

which has infinite no of zeroes. say $\lambda_1, \lambda_2 \ldots \lambda_n \ldots$

$$\alpha = \frac{\lambda_n}{a} \quad n = 1, 2 \ldots$$

Using the principle of superposition we get the solution..

$$\theta(r,t) = \sum_{n=1}^{\infty} A_n \, J_0\left(\frac{\lambda_n}{a} r\right) \exp\left(-\frac{\nu \lambda_n^2}{a^2} t\right)$$

Now using the initial condition we get...

$$\frac{P}{4\mu}(a^2 - r^2) = \sum_{n=1}^{\infty} A_n \, J_0\left(\frac{\lambda_n}{a} r\right)$$

which is a Fourier - Bessel series. Multiplying both sides by $r J_0(\lambda_m \frac{r}{a})$ and integrating from 0 to a and using the orthogonality property ~~we get~~..

$$\cancel{\int_0^a \frac{P}{4\mu}(a^2 - r^2) \, r \, J_0\left(\frac{\lambda_m}{a} r\right) dr = \sum_{n=1}^{\infty} \int_0^a A_n \, r \, J_0\left(\frac{\lambda_n}{a} r\right) J_0\left(\frac{\lambda_m}{a} r\right) dr}$$

~~But we know~~ $\cancel{\int_0^a r \, J_0\left(\frac{\lambda_m}{a} r\right) J_0\left(\frac{\lambda_n}{a} r\right) dr = 0 \text{ when } n \neq m}$

$\cancel{= a^2}$

$$\int_0^a r \, J_0(\lambda_m r) \, J_0(\lambda_n r) \, dr = 0 \quad \text{if } n \neq m$$

$$= \frac{a^2}{2} J_1^2(\lambda_m a) \quad \text{if } n = m$$

we obtain..

$$A_m = \frac{P}{2\mu a^2 J_1^2(\lambda_m)} \int_0^a r (a^2 - r^2) \, J_0\left(\frac{\lambda_m}{a} r\right) dr$$

which on evaluation reduces to.

$$A_m = \frac{2 P a^2}{\lambda_m^3 \mu J_1(\lambda_m)}$$

Hence the required solution is.....

$$\theta(r,t) = \sum_{m=1}^{\infty} \frac{2 P a^2}{\mu \lambda_m^3 \, J_1(\lambda_m)} J_0\left(\frac{\lambda_m}{a} r\right) \exp\left(-\frac{\nu \lambda_m^2}{a^2} t\right)$$

- **Solution of Diffusion equation in spherical co-ordinates :-**

Consider three dimensional diffusion equation..

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

In spherical co-ordinates (r, θ, ϕ, t) it becomes..-

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{--(i)}$$

where $T = T(r, \theta, \phi, t)$

Let us assume separation of variables in the form.

$$T(r, \theta, \phi, t) = R(r). H(\theta). \Phi(\phi). \beta(t)$$

substituting this in (i) we get...

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2 \sin\theta}\frac{1}{H}\frac{d}{d\theta}\left(\sin\theta \frac{dH}{d\theta}\right) + \frac{1}{\Phi r^2 \sin^2\theta}\frac{d^2\Phi}{d\phi^2} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2$$

Thus we get.

$$\frac{d\beta}{dt} + \lambda^2 \alpha \beta = 0 \Rightarrow \beta = c_0 e^{-\alpha\lambda^2 t} \quad \text{--- (ii)}$$

Also. $r^2 \sin^2\theta\left[\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{H r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dH}{d\theta}\right) + \lambda^2\right]$

$$= -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2 \text{ (say)}$$

which gives ...

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$$\Rightarrow \Phi(\phi) = c_1 e^{im\phi} + c_2 e^{-im\phi} \quad \text{---- (iii)}$$

Now, the other seperated equation is ..

$$\frac{1}{R}\left(R'' + \frac{2}{r}R'\right) + \frac{1}{H r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dH}{d\theta}\right) + \lambda^2 = \frac{m^2}{r^2 \sin^2\theta}$$

$$\Rightarrow \frac{r^2}{R}\left(R'' + \frac{2}{r}R'\right) + \lambda^2 r^2 = \frac{m^2}{\sin^2\theta} - \frac{1}{H\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dH}{d\theta}\right) = n(n+1) \text{ (say)}$$

we get ....

$$R'' + \frac{2}{r} R' + \left\{ \lambda^2 - \frac{n(n+1)}{r^2} \right\} R = 0 \quad \cdots (iv)$$

and $$-\frac{1}{H \sin\theta} \left(\sin\theta \frac{d^2H}{d\theta^2} + \cos\theta \frac{dH}{d\theta} \right) + \frac{m^2}{\sin^2\theta} = n(n+1)$$

$$\Rightarrow \frac{d^2H}{d\theta^2} + \cot\theta \frac{dH}{d\theta} + \left\{ n(n+1) - \frac{m^2}{\sin^2\theta} \right\} H = 0 \quad \cdots (v)$$

Let $R = (\lambda r)^{-1/2} \psi(r)$, then equation (iv) becomes..

$$(\lambda r)^{-1/2} \left[\psi''(r) + \frac{1}{r} \psi'(r) + \left\{ \lambda^2 - \frac{(n+1/2)^2}{r^2} \right\} \psi \right] = 0$$

$$\Rightarrow \psi''(r) + \frac{1}{r} \psi'(r) + \left\{ \lambda^2 - \frac{(n+1/2)^2}{r^2} \right\} \psi(r) = 0, \quad \text{since } (\lambda r) \neq 0$$

which is a Bessel's equation of order $(n + 1/2)$, whose solution is...

$$\psi(r) = A J_{n+1/2}(\lambda r) + B Y_{n+1/2}(\lambda r)$$

$$\Rightarrow R(r) = (\lambda r)^{-1/2} \left[A J_{n+1/2}(\lambda r) + B Y_{n+1/2}(\lambda r) \right]$$

where J_n and Y_n are Bessel functions of 1st and 2nd kind respectively.

Now Let $\mu = \cos\theta$

$$\cot\theta = \mu / \sqrt{1-\mu^2}$$

$$\frac{dH}{d\theta} = -\sqrt{1-\mu^2} \frac{dH}{d\mu}$$

$$\frac{d^2H}{d\theta^2} = (1-\mu^2) \frac{d^2H}{d\mu^2} - \mu \frac{dH}{d\mu}$$

substituting these in (v) we get --

$$(1-\mu^2) \frac{d^2H}{d\mu^2} - 2\mu \frac{dH}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2} \right] H = 0$$

which is an Associated Legendre differential equation, whose solution is..

$$H(\theta) = A' P_n^m(\mu) + B' Q_n^m(\mu)$$

where $P_n^m(\mu)$ and $Q_n^m(\mu)$ are Associated Legendre functions of degree n and of order m of 1st and 2nd kind.

$\therefore$ The general solution is...

$$T(r, \theta, \phi, t) = c_0 e^{-\alpha\lambda^2 t}\left[c_1 e^{im\phi} + c_2 e^{-im\phi}\right] . A(\lambda r)^{-1/2} J_{n+1/2}(\lambda r)$$

$$A' P_n^m(\cos\theta)$$

In this general solution.. the functions $Q_n^m(\mu)$ and $(\lambda r)^{-1/2} Y_{n+1/2}(\lambda r)$ are excluded because they have poles at $\mu = \pm 1$ and $r = 0$ respectively

- **Vibration of an infinite string:-**

The one dimensional wave equation is..

$$u_{xx} = \frac{1}{c^2} u_{tt} \quad \text{---(i)}$$

s.t. $u(x,0) = \mu(x)$ & $u_t(x,0) = \nu(x)$

Let $y = ct$.

$$\therefore \frac{\partial u}{\partial t} = c\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial y^2}$$

Thus (i) becomes..

$$u_{xx} = u_{yy} \quad \text{---- (ii)}$$

The λ-quadratic reduces to...

$$\lambda^2 + 0\lambda - 1 = 0$$

$$\Rightarrow \lambda = \pm 1$$

$\therefore$ The corresponding characteristic equations are..

$$\frac{dy}{dx} + 1 = 0 \Rightarrow y + x = c_1$$

$$\frac{dy}{dx} - 1 = 0 \Rightarrow y - x = c_2$$

We choose ξ, η s.t. $\xi = y + x$

$\eta = y - x$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$= u_\xi - u_\eta$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= u_\xi + u_\eta$$

$$u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

substituting this values in (ii) we get . .

$$4u_{\xi\eta} = 0$$

$$\Rightarrow \quad u_{\xi\eta} = 0$$

Integrating we get . . .

$$u(\xi,\eta) = F(\xi) + G(\eta)$$

$$= F(y+x) + G(y-x)$$

$$= f(x+ct) + g(x-ct)$$

Replacing ξ and η, we have the general solution of the wave equation in the form . .

$$u(x,t) = f(x+ct) + g(x-ct)$$

$$u_t(x,t) = c\,(f'(x+ct) - g'(x-ct))$$

from boundary condition $u(x,0) = \mu(x)$ we get . .

$$f(x) + g(x) = \mu(x) \quad \text{--- (iii)}$$

from $u_t(x,0) = \nu(x)$ we get . .

$$c(f'(x) - g'(x)) = \nu(x)$$

$$\Rightarrow f(x) - g(x) = \frac{1}{c}\int_0^x \nu(\xi)\,d\xi \quad \dots \text{(iv)}$$

solving (iii) and (iv) we get . .

$$f(x) = \frac{\mu(x)}{2} + \frac{1}{2c}\int_0^x \nu(\xi)\,d\xi$$

$$g(x) = \frac{\mu(x)}{2} - \frac{1}{2c}\int_0^x \nu(\xi)\,d\xi$$

$\therefore\ u(x,t) = f(x+ct) + g(x-ct)$

$$= \frac{\mu(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} \nu(\xi)\,d\xi + \frac{\mu(x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} \nu(\xi)\,d\xi$$

$$= \frac{1}{2}\left[\mu(x+ct) + \mu(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \nu(\xi)\,d\xi \ .$$

This is known as D'Alembert's solution of one-dimensional wave equation.

[If $\nu = 0$, i.e. if the string is released from rest then the solution is ... $u(x,t) = \frac{1}{2}\left[\mu(x+ct) + \mu(x-ct)\right]$]

- $u_{tt} = c^2 u_{xx}$

  s.t. $u(x,0) = \sin x$

  $u_t(x,0) = 0$

$\Rightarrow\ u_{tt} = c^2 u_{xx}$ ... (i)

Let $y = ct$.

$\therefore\ u_t = c\,u_y$

$u_{tt} = c^2 u_{yy}$

Thus (i) becomes.

$u_{xx} = u_{yy}$ ... (ii)

$\therefore$ The λ - quadratic reduces to.

$$\lambda^2 + 0.\lambda - 1 = 0$$

$$\Rightarrow \lambda = \pm 1$$

$\therefore$ The corresponding characteristic equations are ..

$$\frac{dy}{dx} + 1 = 0 \Rightarrow y + x = c_1$$

$$\frac{dy}{dx} - 1 = 0 \Rightarrow y - x = c_2$$

We choose $\xi,\ \eta$ s.t.

$$\xi = y + x$$
$$\eta = y - x$$

$$u_x = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$= u_\xi - u_\eta$$

$$u_y = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

$$= u_\xi + u_\eta$$

$$\therefore \; u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Substituting this values in (i) we get...

$$4u_{\xi\eta} = 0$$

$$\Rightarrow u_{\xi\eta} = 0$$

Integrating we get..

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

$$= F(y+x) + G(y-x)$$

$$= f(x+ct) + g(x-ct)$$

Replacing ξ, η, we get..

$$\therefore \; u(x,t) = f(x+ct) + g(x-ct)$$

$$u_t(x,t) = c\,[f'(x+ct) - g'(x-ct)]$$

from BCs $u(x,0) = \sin x$ and $u_t(x,0) = 0$ we get.

$$f(x) + g(x) = \sin x \quad \text{--- (iii)}$$

$$f'(x) - g'(x) = 0$$

$$\Rightarrow f(x) - g(x) = c_1 = \text{constant} \quad \text{-- (iv)}$$

Solving (iii) and (iv) we get...

$$f(x) = \frac{1}{2}(\sin x + c_1)$$

$$g(x) = \frac{1}{2}(\sin x - c_1)$$

$$\therefore \; u(x,t) = \frac{1}{2}(\sin(x+ct) + c_1) + \frac{1}{2}(\sin(x-ct) - c_1)$$

$\therefore\ u(x,t) = \sin x \cos ct$

which is the required general solution.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le l,\ t > 0$$

s.t $\quad u(0,t) = 0,\quad u(l,t) = 0,\ t > 0$

$u(x,0) = f(x),\quad u_t(x,0) = g(x)$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \cdots (i)$$

Let the solution be — $u(x,t) = X(x).T(t)$, then..

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k$$

$\therefore\ X'' - kX = 0$

$T'' - c^2 kT = 0$

<u>case - I</u> :- when $k = 0$.

$\therefore\ X'' = 0 \Rightarrow X = c_1 x + c_2$

$T'' = 0 \Rightarrow T = c_3 t + c_4$

Then $u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

using B.C. $u(0,t) = 0$ and $u(l,t) = 0$ we get

$c_1 = c_2 = 0$

$\therefore\ u(x,t) = 0$

which is a trivial solution, but we want non-trivial solution, so we reject case - I

<u>Case - II</u> when $k > 0\ (= \lambda^2)$

$X'' - \lambda^2 X = 0 \Rightarrow X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$T'' - \lambda^2 c^2 T = 0 \Rightarrow T = c_3 e^{\lambda ct} + c_4 e^{-\lambda ct}$.

$$\therefore\ u(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{\lambda ct} + c_4 e^{-\lambda ct})$$

using B.C $u(0,t)=0$ and $u(l,t)=0$ we get..

$$c_1+c_2=0$$

$$c_1 e^{\lambda l}+c_2 e^{-\lambda l}=0 \quad \Rightarrow \quad c_1=c_2=0$$

$\therefore u(x,t)=0$

which is a trivial solution. so we reject case-II

Case-III :-

when $k<0\ (=-\lambda^2)$

$\therefore X''+\lambda^2 X=0 \Rightarrow X=c_1\cos\lambda x+c_2\sin\lambda x$

$T''+c^2\lambda^2 T=0 \Rightarrow T=$ ~~$c_3\sin c\lambda t+c_4$~~ $c_3\cos c\lambda t+c_4\sin c\lambda t$

$\therefore u(x,t)=(c_1\cos\lambda x+c_2\sin\lambda x)(c_3\cos c\lambda t+c_4\sin c\lambda t)$

Now using $u(0,t)=0,\ u(l,t)=0,\ t>0$ we get..

$$c_1=0$$

$$c_2\sin\lambda l=0$$

$\Rightarrow \sin\lambda l=0=\sin n\pi$ [for non-trivial solution $c_2\neq 0$]

$$\Rightarrow \lambda=\frac{n\pi}{l}$$

$$u(x,t)=c_2\sin\frac{n\pi x}{l}\left[c_3\cos\frac{n\pi ct}{l}+c_4\sin\frac{n\pi ct}{l}\right]$$

By the principle of superposition we get..

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left[A_n\cos\frac{n\pi ct}{l}+B_n\sin\frac{n\pi ct}{l}\right]$$

$$u_t(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left[A_n\left(-\frac{n\pi c}{l}\right)\sin\frac{n\pi ct}{l}+B_n\left(\frac{n\pi c}{l}\right)\cos\frac{n\pi ct}{l}\right]$$

Now, $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$

$$\therefore f(x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l}$$

$$g(x)=\sum_{n=1}^{\infty}B_n\frac{n\pi c}{l}\sin\frac{n\pi x}{l}$$

$\therefore \quad A_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$

$\frac{n\pi c}{l} B_n = \frac{2}{l}\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx$

- $u_{tt} = c^2 u_{xx}, \quad 0 \le x \le 2, \quad t > 0$

S.t. $u(0,t) = u(2,t) = 0$

$u(x,0) = \sin 3\frac{\pi x}{2}$

$u_t(x,0) = 0$

$\Rightarrow \quad u_{tt} = c^2 u_{xx} \;\dots\; (i)$

Let the solution be ... $u(x,t) = X(x) \cdot T(t)$, then.

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k$$

$\therefore X'' - kX = 0$

$T'' - c^2 k T = 0$

<u>case-I :-</u> when $k = 0$,

$\therefore X'' = 0 \Rightarrow X = c_1 x + c_2$

$T'' = 0 \Rightarrow T = c_3 t + c_4$

Then $u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

using B.C. $u(0,t) = u(2,t) = 0$, we get --

$c_1 = c_2 = 0$

$\therefore u(x,t) = 0$

which is a trivial solution, but for non-trivial solution we reject case-I !

<u>case-II :-</u> when $k > 0 \; (= \lambda^2)$

$\therefore X'' - \lambda^2 X = 0 \Rightarrow X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$T'' - \lambda^2 c^2 T = 0 \Rightarrow T = c_3 e^{\lambda c t} + c_4 e^{-\lambda c t}$

$\therefore u(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{\lambda c t} + c_4 e^{-\lambda c t})$

using B.C. $u(0,t) = u(2,t) = 0$, we get ..

$c_1 + c_2 = 0$

$c_1 e^{2\lambda} + c_2 e^{-2\lambda} = 0 \quad \Rightarrow \quad c_1 = c_2 = 0$

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$u(x,t) = 0$

is a trivial solution, so we reject case-II.

Case-III:-

$k < 0 \; (= -\lambda^2)$

$X'' + \lambda^2 X = 0 \Rightarrow X = c_1 \cos\lambda x + c_2 \sin\lambda x$

$T'' + \lambda^2 c^2 T = 0 \Rightarrow T = c_3 \cos\lambda ct + c_4 \sin\lambda ct$

$u(x,t) = (c_1 \cos\lambda x + c_2 \sin\lambda x)(c_3 \cos\lambda ct + c_4 \sin\lambda ct)$

using $u(0,t) = u(2,t) = 0$ we get..

$c_1 = 0$

$c_2 \sin 2\lambda = 0$

$\Rightarrow \sin 2\lambda = 0 = \sin n\pi$ [for non-trivial solution $c_2 \neq 0$]

$\Rightarrow 2\lambda = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{2}$

$u(x,t) = c_2 \sin\frac{n\pi x}{2}\left(c_3 \cos\frac{n\pi ct}{2} + c_4 \sin\frac{n\pi ct}{2}\right)$

By the principal of superposition we get...

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{2}\left(A_n \cos\frac{n\pi ct}{2} + B_n \sin\frac{n\pi ct}{2}\right)$$

$$\therefore u_t(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{2}\left[A_n\left(-\frac{n\pi c}{2}\right)\sin\frac{n\pi ct}{2} + B_n\left(\frac{n\pi c}{2}\right)\cos\frac{n\pi ct}{2}\right]$$

$u_t(x,0) = 0$ gives..

$\sum B_n \frac{n\pi c}{2} \sin\frac{n\pi x}{2} = 0$

$B_n = 0$

$\therefore B_n = 0, \; n = 1, 2..$

$u(x,0) = ~~0~~ ~~gives~~ \sin^3\frac{\pi x}{2}$ gives..

$\sum A_n \sin\frac{n\pi x}{2} = \sin^3\frac{\pi x}{2} = \frac{3}{4}\sin\frac{\pi x}{2} - \frac{1}{4}\sin\frac{3\pi x}{2}$

which gives $A_1 = \frac{3}{4}$, $A_3 = -\frac{1}{4}$

while all other A_n's are zero

Hence the required solution is.

$$u(x,t) = \frac{3}{4}\sin\frac{\pi x}{2}\cos\frac{\pi c t}{2} - \frac{1}{4}\sin\frac{3\pi x}{2}\cos\frac{3\pi c t}{2}$$

- **Uniqueness of solution :-**

- The solution of the following problem $u_{tt} = c^2 u_{xx}$, if exists, is unique.

$\Rightarrow$ $u_{tt} - c^2 u_{xx} = F(x,t)$, $0<x<l$, $t>0$

$u(x,0) = f(x) \qquad 0 \le x \le l$

$u_t(x,0) = g(x) \qquad 0 \le x \le l$

$u(0,t) = u(l,t) = 0 \qquad t \geq 0$

Let there be two solutions u_1, u_2 of the above PDE, then $v = u_1 - u_2$ will be a solution of the following problem---

$v_{tt} - c^2 v_{xx} = 0 \qquad 0<x<l$

$v(x,0) = 0$, $0 \le x \le l$

$v_t(x,0) = 0$, $0 \le x \le l$

$v(0,t) = v(l,t) = 0$, $t \geq 0$

$\therefore$ We shall show that ... $v \equiv 0 \Rightarrow u_1 = u_2$

Now, we consider a function $E(t) = \frac{1}{2}\int_0^l (c^2 v_x^2 + v_t^2)\,dx$.

$$\therefore \frac{dE}{dt} = \frac{1}{2}\int_0^l (c^2\, 2 v_x v_{xt} + 2 v_t v_{tt})\,dx$$

$$= \int_0^l (c^2 v_x v_{xt} + v_t v_{tt})\,dx$$

$$= \int_0^l v_t v_{tt}\,dx + \left[c^2 v_x v_t\right]_0^l - \int_0^l c^2 v_t v_{xx}\,dx$$

$$= \int_0^l v_t (v_{tt} - c^2 v_{xx})\,dx$$

$$= 0$$

$\Rightarrow$ E = constant

Given that $V_t(x,0) = 0$ and since $V(x,0) = 0$, we obtain $E(0) = 0$

$\therefore E \equiv 0$

Hence $V_x \equiv 0$ and $V_t \equiv 0$ $\forall\ t > 0$ and $0 < x < l$

This is possible only when $V(x,t)$ is constant but $V(x,0) = 0$

$\therefore V \equiv 0$

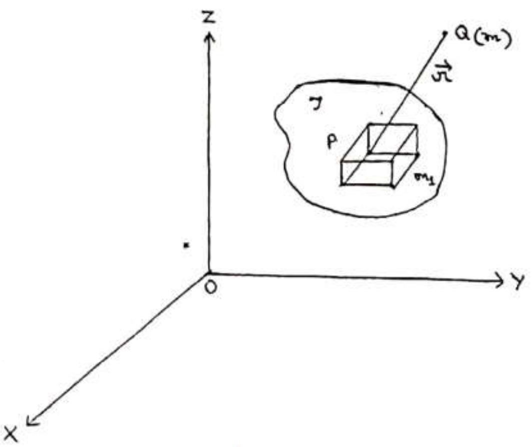
$\Rightarrow u_1 = u_2$

$\therefore$ The solution is unique.

The solution of problem of vibration of a string of finite length is also unique as it is a special case of above problem.

• **Elliptic equation :-**

• **Derivation of laplace equation :-**



Consider two particle of mass m and m_1 situate at Q and P seperated by a distance r as shown in the above figure.

According to the Newton's universal law of gravitation, the magnitude of force is proportional to the product of their masses and inversely proportional to the square of the distance, between them is given by.

$$F = G\frac{m m_1}{r^2} \quad \text{--- (i)}$$

If $\vec{PQ} = \vec{r}$, $m = 1$, $G = 1$, the force at Q due to the mass at P is given by---

$$\vec{F} = -\frac{m_1 \vec{r}}{r^3}$$

$$= \nabla\left(\frac{m_1}{r}\right) \quad \text{...(ii)}$$

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which is called the intensity of the gravitational force.

Suppose a particle of unit mass moves ~~and~~ under the attraction of a particle of mass m_1 at P from infinity up to Q, then the work done by the force F is -

$$\int_{\infty}^{r} \vec{F}\cdot d\vec{r} = \int_{\infty}^{r} \nabla\left(\frac{m_1}{r}\right) dr$$

$$= \frac{m_1}{r} \quad \text{--- (iii)}$$

This is defined as the potential V at Q due to a particle at P and is defined by --

$$V = -\frac{m_1}{r} \quad \text{--- (iv)}$$

From equation (ii), the intensity of the force at P is --

$$\vec{F} = -\nabla V \quad \text{--- (v)}$$

Now, if we consider a system of particles of masses $m_1, m_2, \ldots m_n$ which are at distances $r_1, r_2 \text{---} r_n$, then the force of attraction per unit mass at Q due to the system is --

$$\vec{F_s} = \sum_{i=1}^{n} \nabla \frac{m_i}{r_i} = \nabla \sum_{i=1}^{n} \frac{m_i}{r_i} \quad \text{--- (vi)}$$

$\therefore$ The work done by the force acting on the particle is --

$$\int_{\infty}^{r} \vec{F_s}\cdot d\vec{r} = \sum_{i=1}^{n} \frac{m_i}{r_i} = -V \quad \text{--- (vii)}$$

$$\therefore \nabla^2 V = -\nabla^2 \sum_{i=1}^{n} \frac{m_i}{r_i} = \sum_{i=1}^{n} \nabla^2 \frac{m_i}{r_i} = 0, \quad r_i \neq 0.$$

$\therefore$ ~~$\nabla^2 V = 0$ is called the laplace equation~~

where $\nabla^2 = \cancel{\mathrm{div}}\ \mathrm{div}\, \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the laplace operator

In the case of continuous distribution of matter of density ρ in a volume τ we have ... $V(x,y,z) = \iiint_{\tau} \frac{\rho(\xi,\eta,\zeta)}{r} d\tau$

where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ and Q is outside the body. It can be verified that ...

$$\nabla^2 V = 0$$

which is called the laplace equation.

• Boundary value problems :-

Let R be a bounded region. we shall denote the set of all boundary point of R by ∂R. By the closure of R. we mean the set of all interior points of R together with its boundary points and is denoted by $\bar{R} = R \cup \partial R$

If a function $f \in C^{(n)}$, then all its derivatives of order n are continuous.

~~There are mainly three types of Br~~

There are mainly three types of boundary value problems for Laplace equation. If $f \in C^{(0)}$ and is specified on the boundary ∂R of some finite region R, the problem of determining a function $\psi(x,y,z)$ s.t $\nabla^2\psi = 0$ within R and satisfying $\psi = f$ on ∂R is called the boundary value problem of first kind or the interior Dirichlet problem.

If $f \in C^{(0)}$ and is prescribed on the boundary ∂R of a finite simply connected region R. The problem of determining a function $\psi(x,y,z)$ which satisfy $\nabla^2\psi = 0$ outside R and is s.t $\psi = f$ on ∂R is called an exterior Dirichlet problem.

e.g. Determination of distribution of the potential outside of a body whose surface potential is prescribed is an exterior Dirichlet problem.

The 2nd type Boundary value problem is associated with von Neumann. The problem is to determine the function $\psi(x,y,z)$ s.t. $\nabla^2\psi = 0$ within R while $\partial\psi/\partial n$ is specified at every point of ∂R, where $\partial\psi/\partial n$ denotes the normal derivative of the field variable ψ.

• Some important mathematical results :-

• [illegible] Divergence Theorem :-

Let ∂R be a closed surface in the 3-dimensional space and R denotes the bounded region enclosed by ∂R in which $\vec{F}$ is a vector belonging to $C^{(1)}$ in R and continuous on R. Then.

$$\iint_{\partial R} \vec{F}\cdot\hat{n}\, dS = \iiint_R \nabla\cdot\vec{F}\, dV$$

where dV is an element of volume, dS is an element of surface area and $\hat{n}$ the outward drawn normal.

• Green's Identity :-

Green's 2nd identity :-

$$\iiint_R (\phi \nabla^2\psi - \psi \nabla^2\phi)\, dV = \iint_{\partial R} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) ds .$$

Green's 1st identity :-

$$\iiint_R \nabla\phi \;\nabla\psi \; dV = \iint_{\partial R} \phi \frac{\partial \psi}{\partial n}\, ds - \iiint_R \phi \nabla^2\psi \, dV$$

If $\phi = \psi$, then..

$$\iiint_R (\nabla\phi)^2 dV = \iint_{\partial R} \phi \frac{\partial \phi}{\partial n}\, ds - \iiint_R \phi \nabla^2\phi \, dV$$

• Properties of Harmonic functions :-

Solutions of Laplace equation are called Harmonic functions which possess a number of interesting properties and they are presented in the following theorems.

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I. If a harmonic function vanish everywhere on the boundary then it is identically zero everywhere.

$\Rightarrow$ Let ϕ be a harmonic function, then $\nabla^2\phi = 0$ in R.

Also, if $\phi = 0$ on ∂R, we shall show that $\phi = 0$ in $\bar{R} = R \cup \partial R$.

Recalling Green's 1st identity

$$\iiint_R (\nabla\phi)^2 dV = \iint_{\partial R} \phi \frac{\partial \phi}{\partial n}\, ds - \iiint_R \phi \nabla^2\phi \, dV$$

$$\Rightarrow \iiint_R (\nabla\phi)^2 dV = 0$$

since $(\nabla\phi)^2$ is +ve, it follows that the integral will be satisfied only if $\nabla\phi = 0 \Rightarrow \phi$ is constant in R.

since ϕ is continuous in $\bar{R}$ and ϕ is zero on ∂R,

It follows that $\phi = 0$ in R.