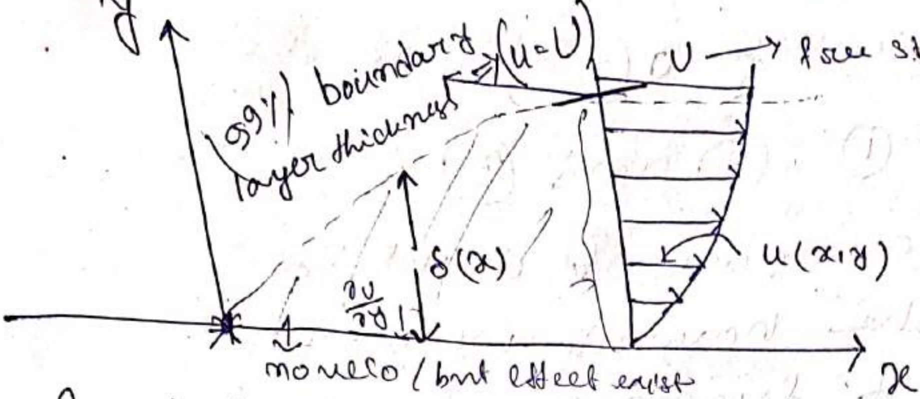
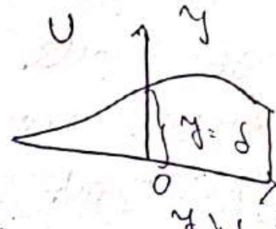
Theory of Boundary Layer : (L. Prandtl 1904)

Viscous ($\mu \neq 0$) :-





Condition :-

$\tau = \mu \frac{\partial u}{\partial y}$ at plate

i) $y = 0$ $\frac{\partial u}{\partial y} > 0$ (frictional), $u = 0$ (no slip condn)

ii) $y = \delta$ $\frac{\partial u}{\partial y} = 0$, $u = U$

* (v.v.I) Two dimensional boundary layer eqns for a flow over a plane wall :- (defn)

Order magnitude approach :-

The Navier stokes eqns in viscous incompressible two dim fluids are -

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{--- (1) along } x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad \text{--- (2) along } y$$

and $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ — (3)

where the xy plane is the plane of motion and x axis along the plate and y axis ⊥r to it. Thus due to no slip condition and since the wall is solid (non-porous) both u and v will vanish at $y = 0$

we will now assess ~~axis~~ of the order of magnitude, symbolically $O(\ )$, of the terms in eqn (1), (2) and (3).

The velocity component u ∥ to the wall in the boundary layer rises rapidly from a velocity zero at the wall to a velocity U in the mainstream within a short distance, δ (say) the thickness of ~~boon~~ boundary layer from the wall. Taking t, x, u as quantities of order 1 (~~$O(1)$~~) and y of $O(\delta)$ where $\delta << 1$

Now we see from eqn of continuity (3)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

we find that $v = O(\delta)$

$\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ each of order 1

$\frac{\partial u}{\partial y}$ $O(\delta^{-1})$

$\frac{\partial^2 u}{\partial y^2}$ $O(\delta^{-2})$

So, eqn(1) $\frac{\partial^2 u}{\partial x^2}$ may be neglected to comparison $\frac{\partial^2 u}{\partial y^2}$, eqn (1) reduces to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

eqn ② reduces to

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 v}{\partial y^2}$$

(δ small) avoid

$-\frac{1}{\rho}\frac{\partial p}{\partial y}$ is of $O(\delta)$

$\therefore$ p increases across the bdd layer of order δ^2 ~~is of $O(\delta^2)$~~

and may be neglected

Hence the pressure is taken practically constant in the direction normal to the boundary layer and may be assumed equal to that at the outer edge of the boundary layer where it is determined by inviscid flow (potential flow).

eqn ① $u = U$ $y = \delta$ to calculate p

$$\boxed{\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \cancel{v\frac{\partial U}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial x}}$$

(condn $\frac{\partial U}{\partial y} = 0$)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

and $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ only x-axis

which is required eqn

Bdd condn at $y = 0$ $u = v = 0$

as $y \to \infty(\delta)$ $u = U(x,t)$

for steady $\frac{\partial u}{\partial t} = 0$ | freestream $U = \text{const}$

$\frac{\partial U}{\partial x} = 0$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

b) Assymtotic approach

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$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

order magnitude approach

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$t^* = \frac{t}{T}$, $x^* = \frac{x}{X}$ (non dimensional)

$y^* = y/Y$ $\quad u^* = \frac{u}{U}$, $v^* = \frac{v}{V}$, $p^* = \frac{p}{P}$

where $t^*, x^*, \dots$ unit of measurement corresponding quantities.

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial(u^* U)}{\partial(t^* T)} = \frac{U}{T}\frac{\partial u^*}{\partial t^*}}$$

(1) becomes

$$\frac{U}{T}\frac{\partial u^*}{\partial t^*} + (u^* U)\cdot\frac{U}{X}\frac{\partial u^*}{\partial t^*} + (v^* V)\cdot\frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho}\frac{P}{X}\left(\frac{\partial p^*}{\partial x^*}\right) + \nu\left[\frac{U}{X^2}\frac{\partial^2 u^*}{\partial x^2} + \frac{V}{Y^2}\frac{\partial^2 u^*}{\partial y^2}\right] \quad (2)$$

$$\left[\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{U}{X}\frac{\partial u^*}{\partial x^*}\right] = \frac{U}{X^2}\frac{\partial^2 u}{\partial x^2}\right]$$

eqn (2) multiplying by X and divided by U^2

$$\frac{X}{UT}\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + \frac{v^* V}{U}\frac{X}{Y}\frac{\partial u^*}{\partial y^*}$$

$$= -\frac{1}{\rho}\frac{P}{U^2}\frac{\partial p^*}{\partial x^*} + \nu\left[\frac{1}{XU}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{X}{UY^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right]$$

non direction motion along x-direction.

along y-direction can be written — Similarly,

$$\frac{X}{UT}\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + \frac{XV}{YU}v^*\frac{\partial v^*}{\partial y^*} = -\frac{PX}{\rho YUV}\frac{\partial p^*}{\partial y^*}$$

$T = \frac{X}{U}$

$$+\frac{\nu}{XU}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\nu X}{Y^2 U}\frac{\partial^2 v^*}{\partial y^{*2}}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{XV}{YU}\frac{\partial v^*}{\partial y^*} = 0 \quad \text{(eqn of continuity)}$$

We shall X and U as the fundamental units an introduced reynolds number of the flow $Re = \frac{UL}{\nu}$

$Re = \frac{UL}{\nu}$

now the unit of time and pressure of X & U can be represented as $T = \frac{X}{U}$ (inte-

& $P = \rho U^2$

we have to rewrite unit measurement of Y & V

$$\boxed{\frac{XV}{YU} = 1} \qquad \boxed{\frac{\nu X}{Y^2 U} = 1}$$

$$y = \frac{x}{\sqrt{Re}} \quad \text{—— (A)}$$

$$v = \frac{U}{\sqrt{Re}} \quad \text{—— (B)}$$

$$\frac{x\nu}{YU} = 1$$

$$y^2 = \frac{\nu x^2}{Ux} = \frac{x^2}{\frac{Ux}{\nu}} = \frac{x^2}{Re}$$

$$v = \frac{y}{x} U = \frac{U}{\sqrt{Re}}$$

reduces to

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re} \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}$$

along y

$$\frac{1}{Re}\left(\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re^2}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{1}{Re}\frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

When Re is large the above set of eqn possess a small parameter and we regard this as $\frac{1}{\sqrt{Re}}$ which is also used (A) & (B) we develope the solution of the above eqn with the help of power series in this small parameter.

$$u^* = u_0 + \frac{1}{\sqrt{Re}} u_1 + \frac{1}{(\sqrt{Re})^2} u_2 + \dots +$$

$$v^* = v_0 + \frac{1}{\sqrt{Re}} v_1 + \frac{1}{(\sqrt{Re})^2} v_2 + \dots$$

$$p^* = p_0 + \frac{1}{\sqrt{Re}} p_1 + \frac{1}{(\sqrt{Re})^2} p_2 + \dots$$

Substitute u^*, v^* & p^* in last 3 eqn

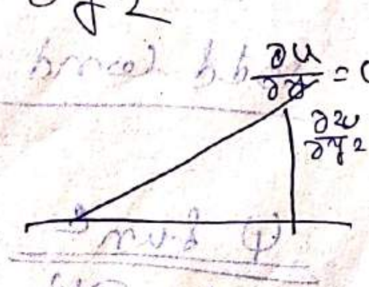
$Re \to \infty$ then term $\frac{1}{\sqrt{Re}} \to 0$

$$\left\{ \begin{aligned} &\frac{\partial u_0}{\partial t^*} + u_0 \frac{\partial u_0}{\partial x^*} + v_0 \frac{\partial u_0}{\partial y^*} = -\frac{\partial p_0}{\partial x^*} + \frac{\partial^2 u_0}{\partial y^{*2}} \\ &-\frac{\partial p_0}{\partial y^*} = 0 \\ &\frac{\partial u_0}{\partial x^*} + \frac{\partial v_0}{\partial y^*} = 0 \end{aligned} \right.$$

reverting to the dimensional form an dropping the index zero well known ream boundary eqn for a plane viscouse unstea incompressible fluid flow are —

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



at the edge of the boundary layer —

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{\partial p}{\partial x}$$

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}\right) + \nu\left(\frac{\partial^2 u}{\partial y^2}\right) \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{cases}$$

which is required eqn.

Bdd cond At $y=0$; $u=0$, $v=0$

as $y \to \infty$ $u = U$

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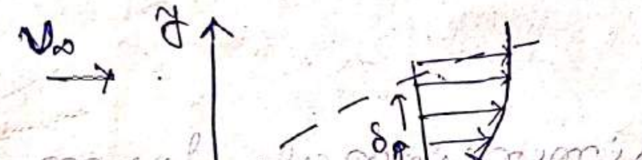
The boundary layer on a flat plate (Blasions solution) :— Page:-170



Bdd eqn are

Bdd layer eqn
$$\begin{cases} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \quad \text{---(1)} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{---(2)} \end{cases}$$

(from (standard bdd layer eqn)

$\left[-\frac{1}{\rho}\frac{\partial p}{\partial x} = U\frac{\partial U}{\partial x} = 0\right]$

Bdd cond $y=0$ $u=0$

$y \to \infty$ $u = U_\infty$

ψ func $u = \frac{\partial \psi}{\partial y}$ $v = -\frac{\partial \psi}{\partial x}$ ——— (3)

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from ①

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} \quad \text{—④}$$

at $y = 0$ $\quad \frac{\partial \psi}{\partial y} = 0$, $\psi = \text{const}$

$y \to \infty$ $\quad \frac{\partial \psi}{\partial y} = U_\infty$ —⑤

$y = 0$

$\frac{\partial \psi}{\partial x} = 0$

$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$

$\psi = \text{const.}$

we take

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{\frac{L}{\sqrt{Re}}} = y\sqrt{\frac{U_\infty}{\nu L}}$$

$$\left[y^* = \frac{y}{\delta} \rightarrow \left[\delta = \frac{L}{(Re)^{\frac{1}{2}}}\right] \right]$$

$Re = \frac{U_\infty L}{\nu}$

from eqn ③ the dimension of ψ is same as the dimension of $U_\infty \frac{L}{\sqrt{Re}} = \sqrt{\nu U_\infty L}$

$$\psi^* = \frac{\psi}{\sqrt{\nu U_\infty L}} = f(x^*, y^*)$$

$\left[u = \frac{\partial \psi}{\partial y},\ uy = \psi,\ U_\infty \delta = \psi \right]$

$$\Rightarrow \psi = \sqrt{\nu U_\infty L}\, f\left(\frac{x}{L}, y\sqrt{\frac{U_\infty}{\nu L}}\right) \quad \text{—⑥}$$

$$\psi = \sqrt{\nu U_\infty L}\sqrt{\frac{x}{L}}\, \Phi\left(\frac{y\sqrt{\frac{U_\infty}{\nu L}}}{\sqrt{x/L}}\right)$$

$$\Rightarrow \psi = \sqrt{\nu U_\infty x}\, \Phi(\eta) \quad \text{where } \boxed{\eta = y\sqrt{\frac{U_\infty}{\nu x}}}$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = U_\infty \Phi'(\eta)$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\{\eta \Phi' - \Phi\}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{U_\infty}{x}\eta \Phi''$$

$$\frac{\partial u}{\partial y} = \ldots$$

(A)

Boundary layer thickness :- page - (176)

(Defn) (δ) $\rightarrow$ bdd layer thickness (defn)

(defn) displacement thickness :- δ_1

$$U\delta_1 = \int_0^\infty (U-u)\,dy$$ bdd layer-u

$$\delta_1 = \frac{1}{U}\int_0^\infty (U-u)\,dy = \int_0^\infty \left(1-\frac{u}{U}\right)dy$$

Mathematically expressions (δ_2)

Thickness
$\delta \rightarrow$ bdd lay
$\delta_1 \rightarrow$ displacement thickness
$\delta_2 \rightarrow$ momentum thickness
$\delta_3 \rightarrow$ energy

(momentum thickness) $$\delta_2 = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

(Energy thickness) $$\delta_3 = \int_0^\infty \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy$$

Momentum thickness (δ_2) :-

The loss of momentum in the boundary layer, as compared with the potential flow, is given by

$$\int_0^\infty \rho u U\,dy - \int_0^\infty \rho u u\,dy$$

$$\Rightarrow \rho\int_0^\infty u(U-u)\,dy$$

and it is denoted by $\rho U^2 \delta_2$

$$\delta_2 = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

Skin friction :- (179)

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

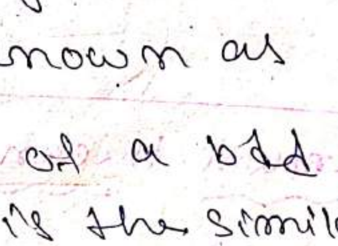
$$= \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}}\ \phi''(0)$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \frac{0.664}{\sqrt{Re_x}}$$

Similar solutions of the bdd layer eqns :— (180)

The solution of boundary layer eqns it depends on the velocity component $\frac{u}{U_\infty}$ or only on one parameter $\eta \left(= \frac{y}{\delta}\right)$ is known as similarity solutions of a bdd layer eqns where η is the similarity variable

Mathematically for similarity solutions it will be possible to transform the partial differential eqns of the bdd layer flow into an ordinary differential eqns.



Bdd layer permitted the similarity soln,

$$\frac{u(x_1, y_1)}{U(x_1)} = \frac{u(x_2, y_2)}{U(x_2)}$$ if it is possible then similarity sol^n exist

Integral methods for the Approximate of the boundary layer eq^{n}s : Page-239

Kármán Momentum Integral eq^{n}s :- / Momentum integral eq^n :-

for steady two dimensional incompressible boundary layer eq^{n}s of the flow are ——— along with the bdd cond^n

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{—(1)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{—(2)}$$

bdd $\quad y = 0, \quad u = v = 0 \quad$ —(3)

$\quad y = \delta \quad u = U(x) \quad$ —(4)

where $\delta(x)$ is the boundary layer thickness

integrating (1) w.r.t y in betn $y=0$ $y=\delta$

$$\int_0^\delta u\frac{\partial u}{\partial x} + \int_0^\delta v\frac{\partial u}{\partial y} = \int_0^\delta U\frac{dU}{dx}\,dy + \int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}\,dy$$

$$= \int_0^\delta U\frac{dU}{dx}\,dy + \nu\int_0^\delta \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \int_0^\delta U\frac{dU}{dx}\,dy - \nu\left(\frac{\partial u}{\partial y}\right)_0 \quad \text{—(5)}$$

$$\left.\frac{\partial u}{\partial y}\right|_0^\delta = \cancel{\left(\frac{\partial u}{\partial y}\right)_\delta} - \left(\frac{\partial u}{\partial y}\right)_0$$

$$\Rightarrow \frac{d}{dx}\int_0^\delta u(U-u)\,dy + \frac{dU}{dx}\int_0^\delta (U-u)\,dy = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$\frac{d}{dx}(U^2\delta_2) + U\frac{dU}{dx}\delta_1 = \frac{\tau_0}{\rho}$$

$$U^2\frac{d\delta_2}{dx} + (2\delta_2+\delta_1)U\frac{dU}{dx} = \frac{\tau_0}{\rho}$$

which is known as Karman integral eqn for two dimensional steady incompressible boundary layer. / Momentum integral eqn

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Energy Integral equations :- (252)

$$\int_0^\delta u^2\frac{\partial u}{\partial x}dy + \int_0^\delta uv\frac{\partial u}{\partial y}dy = U\frac{dU}{dx}\int_0^\delta u\,dy + \int_0^\delta u\frac{\partial^2 u}{\partial y^2}$$ — (1)

$$\int_0^\delta uv\frac{\partial u}{\partial y}dy = v\frac{u^2}{2}\Big|_0^\delta - \frac{1}{2}\int_0^\delta u^2\frac{\partial v}{\partial y}dy$$

$$= v_\delta\frac{U^2}{2} + \frac{1}{2}\int_0^\delta u^2\frac{\partial u}{\partial x}dy$$

$$= -\frac{U^2}{2}\int_0^\delta\frac{\partial u}{\partial x}dy + \frac{1}{2}\int_0^\delta u^2\frac{\partial u}{\partial x}dy$$ — (2)

$$\int_0^\delta u\frac{\partial^2 u}{\partial y^2}dy = u\frac{\partial u}{\partial y}\Big|_0^\delta - \int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2 dy$$

$$= -\int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2 dy$$ — (3)

$$\frac{1}{2}\int_0^\delta\frac{\partial}{\partial x}(u^3)\,dy - \frac{U^2}{2}\int_0^\delta\frac{\partial u}{\partial x}dy = U\frac{dU}{dx}\int_0^\delta u\,dy - \nu\int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2 dy$$ — (4)

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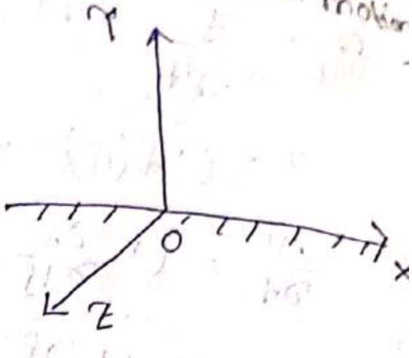
V.V.I 16.53

Problem 1- Unsteady flow of a viscous incompressible fluid over a suddenly accelerated flat plate (flow over a plane wall suddenly set in a motion)

Solution

Let there be an incompressible viscous fluid over a half plane $y=0$ that is xz plane.



viscous

Let the fluid in contact with the plate be infinite in extent and at rest at time $t<0$ at $t=0$ the flat plate suddenly set in a motion with const velo in x-direction, so this generates a two dimensional parallel flow near the flat p

$v=0$, $w=0$, $\frac{\partial}{\partial z}=0$ (no change along z-dir)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \Rightarrow \boxed{\frac{\partial u}{\partial x}=0}$$

$\hookrightarrow u=u(y,t)$

eqn of motion

$$\frac{\partial u}{\partial t}+u\cancel{\frac{\partial u}{\partial x}}^{0}+v\cancel{\frac{\partial u}{\partial y}}^{0}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{0}+\right.$$

Since the plate is situated in infinite flow; the pressure must be constant everywhere

Navier's stoke's eqn

$$\boxed{\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}} \rightarrow (1)$$

initial and bdd cond

$u=0$, $t\leq 0$ $\forall y$

$u=U$ at $y=0$
$\;\;=0$ at $y\to\infty$ $\Big\}$ when $t>0$

$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}$ (parabolic typ)

To obtain the desired soln the partial differential eqn. is 1st reduce to an ordinary differential eqn by an substitution

$$\eta = \frac{y}{2\sqrt{\nu t}} \quad \left(\frac{x}{S}\right) \quad (\text{similarity soln})$$

PDE → ODE $\quad (y, t \to \eta)$ same, same

$$u = U f(\eta)$$

$$\frac{\partial u}{\partial t} = U \frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial t}$$

$$= U \frac{\partial f}{\partial \eta} \left(-\frac{\eta}{2t}\right)$$

$$\boxed{\frac{\partial u}{\partial t} = -\frac{U}{2t} \eta f'(\eta)}$$

$$\frac{\partial u}{\partial y} = U f' \frac{1}{2\sqrt{\nu t}}$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = \frac{U}{4\nu t} f''(\eta)}$$

from (1)

$$-\frac{U}{2t} \eta f'(\eta) = \nu \frac{U}{4\nu t} f''(\eta)$$

$$\boxed{\frac{f''(\eta)}{f'(\eta)} = -2\eta}$$

(1) integrating w.r.t η

$$\log f'(\eta) = -\eta^2 + \log A$$

$$\boxed{f'(\eta) = A e^{-\eta^2}} \Rightarrow \boxed{f(\eta) = A \int_0^\eta e^{-\eta^2} d\eta + B}$$

where A & B are arbitrary const. is to be determine by bdd cond.

$y = 0 \quad \eta = 0 \quad u = U \,/\, f = 1$ $\qquad f(0) = 1$

$y \to \infty \quad \eta \to \infty \quad u = 0 \,/\, f = 0$ $\qquad f(\infty) = 0$

$\cancel{f(\eta)} = \quad 1 = A \int_0^{\eta} d\eta + B$

$B = 1$

$\eta \to \infty$

$$0 = A \int_0^{\infty} e^{-\eta^2} d\eta + B$$

$$0 = A \frac{\sqrt{\pi}}{2} + B \qquad \boxed{B = -\frac{A\sqrt{\pi}}{2}}$$

$$f(\eta) = A \int_0^{\eta} e^{-\eta^2} d\eta - \frac{A\sqrt{\pi}}{2}$$

$$= A \left[\int_0^{\eta} e^{-\eta^2} d\eta - \frac{\sqrt{\pi}}{2} \right]$$

$\eta = 0$

$$1 = A \left[-\frac{\sqrt{\pi}}{2} \right]$$

$$\boxed{A = -\frac{2}{\sqrt{\pi}}}$$

we have

$$f(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$ (non-dimensional velocity $f = \frac{u}{U}$)

$$u = U \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta \right]$$

$$u = U \left[1 - erf(\eta) \right]$$

Skin friction

$$\tau_{xy} = \mu \frac{\partial u}{\partial y} \Big|_{y=0} = \qquad \frac{\partial u}{\partial y} = U f'(\eta) \frac{1}{2\sqrt{\nu t}}$$

$$\boxed{\tau_{xy} = \mu U f'(0) \frac{1}{2\sqrt{\nu t}}}$$

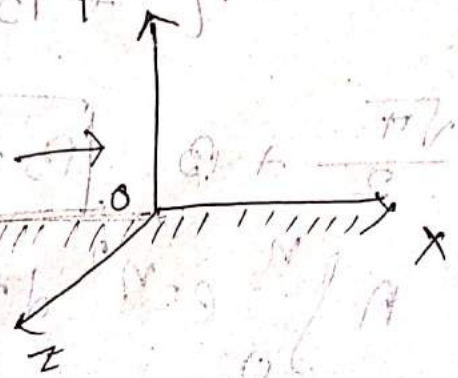
$\boxed{\text{Problem-2}}$ 10

16.58

initial condn

$\boxed{\begin{array}{ll} u = U\cos\omega t & \text{at } y=0 \\ u = 0 & \text{at } y\to\infty \end{array}}$

Unsteady flow of viscous incompressible fluid over a oscillating plate :-



Add

$v=0 \quad w=0 \quad \frac{\partial}{\partial t}=0$

$\frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y,t)$

$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$

at $y=0 \quad u = U\cos nt$

$y\to\infty \quad u=0$

$$u(y,t) = R\left\{ e^{int} \underbrace{f(y)}_{f(\cos nt)} \right\}$$

$$R\left\{ f(y) \times in\, e^{int} \right\} = R\left\{ \nu e^{int} \frac{d^2 f}{dy^2} \right\}$$

$$\boxed{\frac{d^2 f}{dy^2} - \frac{in}{\nu} f = 0}$$

$$f(y) = A e^{(1+i) y\sqrt{\frac{n}{2\nu}}} + B e^{-(1+i) y\sqrt{\frac{n}{2\nu}}}$$

Since $u=0$ at $y\to\infty$

$A=0$

$$\boxed{\begin{array}{l} D^2 = \frac{in}{\nu} \\ D = \pm i^{\frac{1}{2}} \sqrt{\frac{n}{\nu}} \end{array}}$$

$$f(y) = B e^{-(1+i) y\sqrt{\frac{n}{2\nu}}}$$

$$\boxed{B = f(0) = U}$$

$U\cos nt = f(0)\cos nt$

$f(0) = U$

$f(y) = U e^{-(1+i)y\sqrt{\frac{n}{2\nu}}}$

$u(y,t) = R\left\{ e^{int}\, U e^{-(1+i)y\sqrt{\frac{n}{2\nu}}} \right\}$

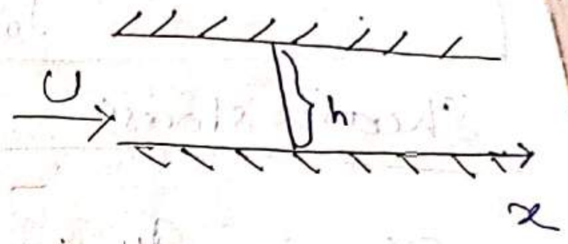
$u(y,t) = U e^{-y\sqrt{\frac{n}{2\nu}}} \cos\left(nt - y\sqrt{\frac{n}{2\nu}}\right)$

Skin friction

16.18

v.v.I Problem 3:- Unsteady flow between two || plates.
(similar 1st problem)

$v=0 \quad w=0 \quad \frac{\partial}{\partial z}=0$



$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$

~~t=0 y~~

$u=0 \qquad t \le 0$

$\left.\begin{aligned} u &= U \quad \text{at } y=0 \\ &= 0 \quad \text{at } y=h \end{aligned}\right\} t > 0$

$\eta = \frac{y}{2\sqrt{\nu t}}$

$u = U f(\eta) \qquad \frac{f''(\eta)}{f'(\eta)} = -2\eta$

$f'(\eta) = A e^{-\eta^2}$

$\boxed{f(\eta) = A\int_0^\eta e^{-\eta^2} d\eta + B}$

at $y=0 \quad \eta=0 \quad u=U \;/\; f(0)=1$

$y=h \quad \eta=\eta_0 \;/\; f(\eta_0)=0$

$\eta=0 \longrightarrow f(0) = B = 1$

$f(\eta) = 1 + A\int_0^\eta e^{-\eta^2} d\eta$

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$$0 = 1 + A \int_0^{\eta_0} e^{-\eta^2} d\eta$$

$$A = -\frac{1}{\int_0^{\eta_0} e^{-\eta^2} d\eta}$$

$$\boxed{f(\eta) = 1 - \frac{1}{\int_0^{\eta_0} e^{-\eta^2} d\eta} \int_0^{\eta} e^{-\eta^2} d\eta}$$

Sheare stress — $y=0 \quad \eta=0$
$y=h \quad \eta=\eta_0$

$$\tau = \mu \frac{\partial u}{\partial y}$$

1) Momentum
2) Energy
3) order
4) Assymtotic
5) 3 meth

defn — 4th April.

Boundary problem final

Page :- 18.46 / 18.20 (Some examples relation betn thickness)

(15-20)

(1, 2, 3, 4, 5, 6, 7, 8, 10, 9) (1-10)
11, 13, 14, 15, 19

ge:- 18.68 (Excercise)

Von Karman - Pohlhausen method :-

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

$$\delta_2 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \qquad \frac{u}{U} = y^2$$

$$\delta_3 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy$$

The similar solution of the boundary layer eqn reveal that the velocity distribution in the boundary layer is some func of the ratio y/δ, δ being boundary layer thickness.

Taking the advances of this functional relationship Pohlhausen assume that even for a general problem of two dimensional boundary layer flow velocity distribution may be taken as some func of $\frac{y}{\delta}$ which he approximated by a polynomial of 4th degree in $\frac{y}{\delta}$ as follows.

$$\boxed{\frac{u}{U} = f(\eta), \quad \eta = y/\delta}$$

$$\frac{u}{U} = f(\eta) = \sum_{i=0}^{4} a_i \eta^i , \quad 0 \le \eta \le 1$$

In order to determine co-efficients a_0, a_1, a_2, a_3, a_4 pohlhausen describes the following boundary & compatibility condition —

$y = 0$: $u = 0$ $\quad \nu\left(\frac{\partial^2 u}{\partial y^2}\right)_0 = -U \frac{dU}{dx}$

$y = \delta$: $u = U$, $\left(\frac{\partial u}{\partial y}\right)_\delta = 0$, $\left(\frac{\partial^2 u}{\partial y^2}\right) = 0$

The 1st condition at $y=0$ usual noslip condition whereas 2nd condn is known as compatibility condn at the surface flows from the boundary layer eqn at the wall surface.

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The condn at $y = \delta$ follows from the consideration at the outer edge of the bdd layer the velocity u in the bdd layer passes smoothly to the potential flow velocity U.

The co-efficient a_0 to a_4 are obtained by the above conditions are —

$$a_0 = 0$$
$$a_1 = 2 + \frac{A}{6}$$
$$a_2 = -\frac{A}{2} \quad \text{where } A = \frac{\delta^2}{\nu}\frac{dU}{dx} \text{ (Shape factor)}$$
$$a_3 = -2 + \frac{A}{2}$$
$$a_4 = 1 - \frac{A}{6}$$

Hence the 4th degree velocity profile ~~at the~~ which satisfy all the bdy conditions is

$$\frac{u}{U} = f(\eta) = F(\eta) + A\,G(\eta) \quad (\mathrm{A})$$

where $F(\eta) = 2\eta - 2\eta^3 + \eta^4$

and $G(\eta) = \frac{\eta}{6}(1-\eta)^3$

The velocity profile is a one parameter family of curve with parameter A known as Shape factor because the shape of the velo. profile ploted against η depends on the value of A.

$$\frac{u}{U} = \sum a_i \eta^i$$

$f(\eta) = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4$

$f' = a_1 + 2a_2\eta + 3a_3\eta^2 + 4a_4\eta^3$

$f'' = 2a_2 + 6a_3\eta + 12a_4\eta^2$

$$\frac{\partial u}{\partial y} = U f' \cdot \frac{1}{\delta}$$

$$\frac{\partial^2 u}{\partial y^2} = U f'' \cdot \frac{1}{\delta^2}$$

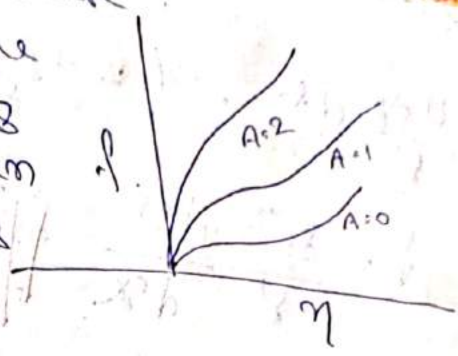
$$\nu U f''(0)\frac{1}{\delta^2} = -U\frac{dU}{dx}$$

$$\frac{\nu U}{\delta^2} \cdot 2a_2 = -U\frac{dU}{dx}$$

$$a_2 = -\frac{\delta^2}{2\nu}\frac{dU}{dx} = -\frac{A}{2}$$

use condn

We are now the position to calculate the value of $\delta(x)$ from the karman momentum integral eqn with the help of pohlhausen velocity profile. For this we first calculate δ_1, δ_2 & τ_0



$[-12 \le A \le 12]$

$$\boxed{U^2 \frac{d\delta_2}{dx} + (2\delta_2 + \delta_1) U \frac{dU}{dx} = \frac{\tau_0}{\rho}}$$

→ karman momentum Integral eqn

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

$$= \int_0^\delta \left[1 - F(\eta) - A\, G(\eta)\right] dy$$

$\eta = \frac{y}{\delta}$

$\delta\, d\eta = dy$

$$= \delta \int_0^\delta \left[1 - F(\eta) - A\, G(\eta)\right] d\eta$$

$$= \delta \left[\frac{3}{10} - \frac{A}{120}\right]$$

Similarly,

$$\delta_2 = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \delta \int_0^\delta \left[F(\eta) + A\, G(\eta)\right]\left[1 - F(\eta) - A\, G(\eta)\right] d\eta$$

$$= \delta \left[\frac{37}{315} - \frac{A}{945} - \frac{A^2}{1072}\right]$$

$$\tau_0 = \mu \frac{\partial u}{\partial y}\Big|_{y=0}$$

$$= \mu \frac{U}{\delta} \frac{\partial f}{\partial \eta}\Big|_{\eta=0}$$

$$= \mu \frac{U}{\delta} \left(2 + \frac{A}{6}\right)$$

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Application of karman-pohlhausen method :-

Case-1 Boundary layer over a flat plate :-

Sol^n let, $U(x)$ = const

$$A = 0 \qquad \left[\because A = \frac{\delta^2}{\nu}\frac{dU}{dx}\right]$$

$$\lambda = 0 \qquad \left[\lambda = \frac{\delta_2^2}{\nu}\frac{dU}{dx}\right]$$

$$\frac{d}{dx}\left[\frac{\delta_2^2}{\nu}\right] = \frac{L(0)}{U} \qquad \left[L(\lambda) = 2\left[\Omega - \lambda(H+2)\right]\right]$$

$$= \frac{2\Omega}{U} \qquad \left[\left(2+\frac{A}{6}\right)\left[\frac{37}{315} - \frac{A}{945} - \frac{A^2}{9072}\right]\right]$$

$$= \frac{2}{U}\left[2\times\frac{37}{315}\right]$$

$$= \frac{4\times 37}{315U} = \frac{148}{315U} = \frac{0.4698}{U}$$

$$\boxed{\delta_2 = 0.685\sqrt{\frac{\nu x}{U}}}$$

$$\boxed{\delta_1 = 1.75\sqrt{\frac{\nu x}{U}}}$$

$$\tau_0 = 0.343\,\mu U\sqrt{\frac{U}{\nu x}}$$

1) Flow in a tube of elliptic cross-section

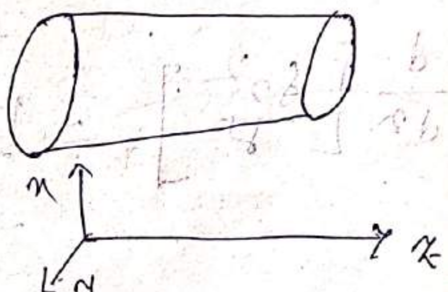
$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right)$:— 16.14

Consider the steady flow of a viscous incompressible fluid through a tube of elliptic cross-section. Let z axis be ∥ to its generators of the tube. The only non-zero component of velocity is the velocity along z axis.

Therefore the eqn of continuity

hence $\boxed{u = v = 0}$

$\boxed{\frac{\partial w}{\partial z} = 0}$ $w = w(x, y)$



i.e. w is a func of x & y only.

The eqn of motion in Cartesian co-ordinates in the absence of external forces reduces to —

i) $\frac{\partial p}{\partial x} = 0$ along x —(1)

$\frac{\partial p}{\partial y} = 0$ " y —(2)

i.e. $\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)\right)$

$\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu(\quad)\right]$

$\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial p}{\partial z}$ along z —(3)

∴ p is fun of z only from (1) & (2)

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{dp}{dz} \cdot \frac{1}{\mu}$$

$$= -\frac{P}{\mu}$$

$\frac{dp}{dz} = -P$

with the boundary condition
$w = 0$ on the surface of tube

$$w = \varphi - \frac{P}{4\mu}(x^2+y^2)$$

$$\boxed{\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0}$$

with the boundary condition $\varphi = \frac{P}{4\mu}(x^2+y^2)$

The suitable solutions of the Laplace equation

$$\varphi = A(x^2-y^2) + B$$

on bdd condition

$$\frac{P}{4\mu}(x^2+y^2) = A(x^2-y^2) + B$$

This will be given ellipse if

$$x^2\left(A - \frac{P}{4\mu}\right) + \left(-A - \frac{P}{4\mu}\right)y^2 = -B$$

$$\Rightarrow \quad a^2\left(\frac{P}{4\mu} - A\right) = b^2\left(\frac{P}{4\mu} + A\right) = B$$

$$A = \frac{P}{4\mu}\left(\frac{a^2-b^2}{a^2+b^2}\right)$$

$$B = \frac{P}{2\mu} \frac{a^2b^2}{a^2+b^2}$$

Hence the velocity distribution in the elliptic cylinder is given by

$$w = \frac{P}{2\mu} \frac{a^2b^2}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

volumetric flow

$$Q = \iint w \, dx \, dy = \pi P \frac{a^3b^3}{4\mu}(a^2+b^2)$$

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Quidelocal triangle / rectangle 16/14 16/15

Energy eqn :-

$$\boxed{\rho C_v \frac{DT}{Dt} = \frac{\partial q}{\partial t} + k \frac{\partial}{\partial x_i}\left(\frac{\partial T}{\partial x_i}\right) + \phi}$$

Page 35

Conservation of Energy :-

Eqn of Energy :-

[Solid → ~~fluid~~ solid]

rate of energy loss by heat conduction

$$\underbrace{\int_v \frac{\partial Q}{\partial t} dv}_{} + \underbrace{\int v_i(\sigma_{ij} n_j) dS}_{} - \int_j q_j n_j \, dS - \int_S E_t \rho v_j n_j = \underbrace{\frac{\partial}{\partial t} \int_v \rho E_t \, dv}_{}$$

the rate at which heat produced by external egency

rate at which heat produced by the work of surface stress

rate of increase of energy in the volume

Connecti (velo. Solid. Shi

radiation - radiative nature

E_t = total energy of the system per unit mass

$$= \frac{1}{2} v_i \cdot v_i + k + I$$

K.E   potential   internal

$$q = -k \frac{\partial T}{\partial x}$$

$$\vec{q} = -k \nabla T$$

Conductivity of the solid

$$q_j = -k \frac{\partial T}{\partial x_j}$$

$$\Rightarrow \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x_j}(v_i \sigma_{ij}) - \frac{\partial}{\partial x_j}(E_t \rho v_j)$$

$$+ \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) - \frac{\partial}{\partial t}(E_t \rho) = 0 \quad \text{--- (1)}$$

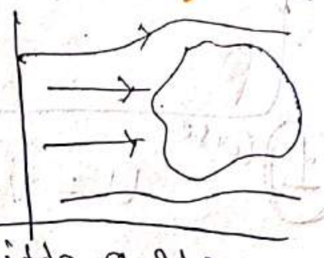
Motion of elliptic cylinder :-

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad w = \phi + i\psi$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad \nabla^2 \phi = 0 = \nabla^2 \psi$$

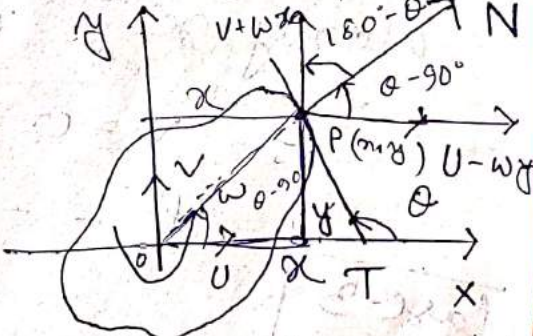
The Conditions :- (i) fluid is at rest at the infinity

$$\frac{\partial \psi}{\partial x} = 0 \quad \frac{\partial \psi}{\partial y} = 0$$



(ii) At any fixed boundary the normal velocity must be zero of the boundary must co-inside with a stream line $\psi = $ Const

(iii) At the boundary moving cylinder the normal component of velocity of the fluid must be equal to the normal component of velocity to the cylinder.



Let a point O taken as origin

$\frac{v}{r}$

due to the rotation

velo. at P $U - \omega y$, $V + \omega x$ | $V = \omega r$

$\cos\theta = \frac{dx}{ds}$ $\sin\theta = \frac{dy}{ds}$

outward normal velocity at P

$$(U - \omega y)\cos(\theta - 90) + (V + \omega x)\cos(180 - \theta)$$

$$= (U - \omega y)\sin\theta - (V + \omega x)\cos\theta$$

$$= (U - \omega y)\frac{dy}{ds} - (V + \omega x)\frac{dx}{ds}$$

$\pi - \theta$

$$-\frac{\partial \psi}{\partial s} = (U - \omega y)\frac{dy}{ds} - (V + \omega x)\frac{dx}{ds}$$

integrating we get -

$$\boxed{\psi = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + C}$$

$$\frac{d\psi}{dx} = \frac{d\psi}{ds}\cdot\frac{ds}{dx}$$

$$ds^2 = dx^2 + dy^2$$

(1) Case

$$U = V = 0$$

$$\psi = \frac{1}{2}\omega(x^2 + y^2) + C$$

$$= \frac{1}{2}\omega z\bar{z} + C$$

Case-2 - if the cylinder moving along x some velo without rotation

$$V = 0, \quad \omega = 0$$

$$\psi = -Uy + C$$

Case-3

$$U = 0, \ \omega = 0$$

$$\psi = Vx + C$$

(a) The kinetic energy of a moving cylinder in a fluid at rest at infinity is given by —

$$T = -\frac{1}{2}\rho \iint_S \phi \frac{\partial \phi}{\partial n} ds$$ Page-7.3

ds is the elementary surface around a closed surface S.

$$\int (\nabla\phi' \cdot \nabla\phi) dv = -\int_S \phi' \frac{\partial \phi}{\partial n} ds - \int_S \phi' \nabla^2 \phi dv$$

$$\int (\nabla \cdot \nabla\phi') dv = -\int_S \phi \frac{\partial \phi'}{\partial n} ds - \int_S \phi \nabla^2 \phi' dv$$

[green's identity]

if $\phi = \phi'$

$$\int (\nabla\phi \cdot \nabla\phi) dv = -\int \phi \frac{\partial \phi}{\partial n} ds$$

[$\because \phi$ is velocity potential then it satisfied $\nabla^2\phi' = 0 \Rightarrow \nabla^2\phi = 0$]

$$\Rightarrow \int (\nabla\phi \cdot \nabla\phi) dv = -\int_S \phi \frac{\partial \phi}{\partial n} ds$$

$$\Rightarrow \int_V \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] dv = -\int_S \phi \frac{\partial \phi}{\partial n} ds$$

Then

$$\Rightarrow \frac{1}{2}\rho \int_V q^2 dv = -\frac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n} ds$$

Motion of elliptic cylinder :-
(do provisions)

$z = c\cosh\zeta$

$x + iy = c\cosh(\xi + i\eta)$

$x = c\cosh\xi\cos\eta$

$y = c\sinh\xi\cos\eta$

where $z = x+iy$, $\zeta = \xi + i\eta$

Hence $\xi = \alpha$

$$\frac{x^2}{c^2\cosh^2\xi} + \frac{y^2}{c^2\sinh^2\xi} = 1$$

$a = c\cosh\alpha$

$b = c\sinh\alpha$ $\quad a^2 - b^2 = c^2$

$a+b = c(\cosh\alpha + \sinh\alpha) = ce^{\alpha}$

$a-b = c(\cosh\alpha - \sinh\alpha) = ce^{-\alpha}$

$$\boxed{e^{2\alpha} = \frac{a+b}{a-b} \quad \alpha = \frac{1}{2}\log\frac{a+b}{a-b}}$$

$z = c\cosh\zeta$, $n\omega y = e^{\alpha}$

The equations $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} = 0$$

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0$$

Solution of this eqn in the form

$$\left.\begin{matrix}\cosh \\ \sinh \\ \exp\end{matrix}\right\} (n\mu) \times \left.\begin{matrix}\cos n\eta \\ \sin\end{matrix}\right\} (n\eta)$$

1) To determine the velocity potential when the elliptic cylinder moves in an infinite fluid with velocity U ||^{el} to the axial plane through the major axis of the cross section. [Page-7.29]

Solution:-

$$\psi = Vx - Uy + \frac{1}{2}\omega(x^2+y^2) + c$$

$$V = 0, \ \omega = 0$$

$$\boxed{\psi = -Uy + A}$$

let the cross section be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\mu = \alpha$$

$$x = c\cosh\alpha\cos\eta \qquad \psi = -Uc\sinh\alpha\sin\eta + A$$

$$y = c\sinh\alpha\sin\eta$$

Since ψ contains $\sin\eta$ at the fluid is rest at infinity ψ must be of the form $e^{-\mu}\sin\eta$

$$\phi + i\psi = Be^{-(\mu+i\eta)}$$

$$\boxed{\psi = -Be^{-(\mu)}\sin\eta}$$

$$-Be^{-\alpha}\sin\eta = -Uc\sinh\alpha\sin\eta + A$$

$$A = 0 \qquad B = Uce^{\alpha}\sinh\alpha$$

$$\psi = -Uc\, e^{-\alpha - \eta} \sinh \alpha \sin \eta$$

Stream func which makes the boundary of the ellipse cylinder

as a stream line when the cylinder moves with velo. U.

$$\psi = -Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\eta} \sin \eta$$

which is rest at infinity $\left[\because e^{2\alpha} = \frac{a+b}{a-b}\right]$

$$\phi = Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\eta} \cos \eta$$

$$w = \phi + i\psi$$

$$= Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\frac{\eta + i\eta}{\eta}}$$

Complex potential... in terms of

7.32

Problem.

Determine ϕ & ψ & w when an elliptic cylinder is rotating with angular velocity ω in an infinite mass of fluid rest at infinity.