

Boundary layer theory

1. Prandtl's boundary layer theory.

The method of dividing the fluid in two regions was proposed by Prandtl in 1904. He suggested that the entire field of flow can be divided for the sake of mathematical analysis, into the following regions:

- (i) A very thin layer (boundary layer) in the vicinity of the plate in which the velocity gradient normal to the wall (i.e., $\partial u / \partial y$) is very large. Accordingly the viscous stress $\mu(\partial u / \partial y)$ becomes important even if μ is small. Thus the viscous and ~~inter~~ inertial forces are of the same order within the boundary layer.
- (ii) In the remaining region (i.e., outside the boundary layer) $\partial u / \partial y$ is very ~~large~~ small & so the viscous forces may be ignored completely. Outside the boundary layer, the flow can be regarded non-viscous and hence the theory of non-viscous fluids offers a very good approximation.

there.

* The following three conditions must be satisfied by any ~~de~~ velocity distribution in boundary layer:

(i) At $y=0$, $u=0$, dy/dy has some finite value

(ii) At $y=\delta$, $u=U$ (iii) At $y=\delta$, $\frac{du}{dy} = 0$

Some definition

1. Boundary layer thickness:

In a qualitative manner, the boundary layer thickness is defined as the elevation above the boundary which covers a region of flow where there is a large velocity gradient and consequently non-negligible viscous effects. Since transition from velocity in the boundary to that outside it takes place asymptotically, there is no obvious demarcation for permitting the measurement of a boundary layer thickness is a simple quantitative manner. For mathematical convenience, the thickness of the boundary layer is generally defined as that distance from the solid boundary where the velocity differs by 1 per cent from the external velocity

U (free flow velocity)

It is easily seen that the above definition of the boundary layer is to a certain extent arbitrary.

(i) Displacement thickness

Because of viscosity the velocity in the vicinity of the plate is smaller than the free-flow region. The reduction in total flow rate caused by this action is

$$\int_0^{\infty} (U - u) dy$$

If this integral is equated to a quantity $U \delta_1$, δ_1 can be considered as the amount by which the potential flow has been displaced from the plate. Thus, for displacement thickness δ_1 , we have the definition

$$U \delta_1 = \int_0^{\infty} (U - u) dy$$

$$\delta_{1,1} = \int_0^{\infty} \left(1 - \frac{u}{U}\right) dy$$

(ii) Momentum thickness

It is defined by comparing the loss of momentum due to wall friction in the boundary to the

momentum in the free flow region. Thus for the momentum thickness δ_2 we have the definition

$$\rho U^2 \delta_2 = \rho \int_0^{\infty} u(U-u) dy$$

(iii) Energy thickness or dissipation energy thickness or kinetic energy thickness.

There is always a loss in energy because of the viscosity of the fluid. Now the loss of kinetic energy in the boundary layer at a distance y from the plate is $\frac{\rho}{2} (U^2 - u^2)$ and consequently the total rate at which the kinetic energy is being lost is

$$\frac{1}{2} \rho \int_0^{\infty} (U^2 - u^2) dy$$

If the integral is equated to a quantity $\frac{\rho}{2} (U^3 \delta_3)$ δ_3 can be considered as thickness of a layer which has the same kinetic energy flux as the rate at which the kinetic energy is being lost in the boundary region. Thus for the energy thickness δ_3 we have the definition on:

$$\frac{1}{2} \rho U^3 \delta_3 = \frac{1}{2} \rho \int_0^{\infty} (U^2 - u^2) dy$$

$$\delta_3 = \int_0^{\infty} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

The boundary layer equations in two-dimensional flow

Here we use asymptotic approach.

For simplicity we shall first derive the boundary layer equations for the flow over a semi-infinite flat plate. We take rectangular cartesian coordinates (x, y) with x measured in the plane in the direction of the two-dimensional laminar incompressible flow, and y measured normal to the plate, and (u, v) are the velocity components.

In what follows, we shall prove that Prandtl boundary layer equations may be looked as the asymptotic form of the Navier-Stokes equations at large Reynold's number.

Let t', x', y', u', v' and p' be dimensionless quantities such that

$$t' = \frac{t}{T}, \quad x' = \frac{x}{X}, \quad y' = \frac{y}{Y}, \quad u' = \frac{u}{U}, \quad v' = \frac{v}{V}$$
$$p' = \frac{p}{P} \quad \text{where } T, X, Y, U, V \text{ are the}$$

units of measurement of the corresponding quantities.

The Navier-Stokes equations of motion for a viscous incompressible fluid in the absence of body forces, in two-dimen-

and flow are given by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (3)$$

$$\gamma = \frac{\mu}{\rho}$$

the equations of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4)$$

Here the xy -plane is the plane of motion, the x -axis is along the wall of the plate and y -axis perpendicular to it. Substituting (1) in equations (2), (3) and (4) we get

$$\frac{U}{\tau} \frac{\partial u'}{\partial x'} + U u' \frac{U}{X} \frac{\partial u'}{\partial x'} + v' V \frac{U}{Y} \frac{\partial u'}{\partial y'} = -\frac{P}{\rho X} \frac{\partial p'}{\partial x'} + \frac{U \nu}{X} \frac{\partial^2 u'}{\partial x'^2} + \frac{\nu U}{Y^2} \frac{\partial^2 u'}{\partial y'^2}$$

$$\Rightarrow \frac{X}{U \tau} \frac{\partial u'}{\partial x'} + u' \frac{\partial u'}{\partial x'} + \frac{X v'}{Y U} \frac{\partial u'}{\partial y'} = -\frac{P}{\rho U^2} \frac{\partial p'}{\partial x'} + \frac{\nu}{X U} \frac{\partial^2 u'}{\partial x'^2} + \frac{\nu X}{Y^2 U} \frac{\partial^2 u'}{\partial y'^2} \quad (5)$$

$$\frac{X}{U \tau} \frac{\partial v'}{\partial x'} + u' \frac{\partial v'}{\partial x'} + \frac{X v'}{Y U} \frac{\partial v'}{\partial y'} = -\frac{P}{\rho Y U^2} \frac{\partial p'}{\partial y'} + \frac{\nu}{X U} \frac{\partial^2 v'}{\partial x'^2} + \frac{\nu X}{Y^2 U} \frac{\partial^2 v'}{\partial y'^2}$$

and $\frac{\partial u'}{\partial n'} + \frac{XV}{YU} \frac{\partial u'}{\partial y'} = 0$ — (4)

Taking X and U as the fundamental units and introduce the Reynolds number of the flow as

$$Re = \frac{XU}{\nu} \quad \text{--- (5)}$$

Then the units of time and pressure can be expressed in terms of X and U as follows:

$$T = \frac{X}{U} \quad \text{and} \quad P = \rho U^2 \quad \text{--- (6)}$$

We now derive the units of measurement of Y and ν by applying the condition that the system of equations (5), (6) and (7) must possess only a single flow parameter Re (given by (5)). Accordingly we must have

$$\frac{XU}{YU} = 1 \quad \text{and} \quad \frac{\nu X}{Y^2 U} = 1 \Rightarrow Y^2 = \frac{\nu X}{U}$$

$$Y^2 = \frac{\nu X^2}{\frac{XU}{\nu}}$$

$$Y = \frac{YU}{X} = \frac{\nu U}{\sqrt{Re}}$$

$$Y = \frac{X}{\sqrt{Re}}$$

Hence (5), (6) & (7) may be rewritten as — (10)

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} = - \frac{\partial p'}{\partial x'} + \frac{1}{Re} \frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} \quad (11)$$

$$\frac{1}{Re} \left(\frac{\partial v'}{\partial t'} + u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} \right) = - \frac{\partial p'}{\partial y'} + \frac{1}{Re^2} \frac{\partial^2 v'}{\partial x'^2} +$$

$$\frac{1}{Re} \frac{\partial^2 v'}{\partial y'^2} \quad (12)$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \quad (13)$$

$$\frac{p\lambda}{\rho\gamma UV} = \frac{\rho U^2 \lambda}{\rho\gamma UV} = \frac{UX}{\gamma V} = \frac{Re \cdot \nu}{\frac{\rho \lambda^2}{\rho} \cdot \frac{U}{\lambda}} = Re$$

If Re is large, then $1/\sqrt{Re}$ can be treated as small parameter.

We now obtain the solution of (11) (12) (13) with help of a power series containing $1/Re$ as follows:

$$\begin{aligned} u' &= u_0 + \left(\frac{1}{\sqrt{Re}}\right) u_1 + \left(\frac{1}{\sqrt{Re}}\right)^2 u_2 + \dots \\ v' &= v_0 + \left(\frac{1}{\sqrt{Re}}\right) v_1 + \left(\frac{1}{\sqrt{Re}}\right)^2 v_2 + \dots \\ p' &= p_0 + \left(\frac{1}{\sqrt{Re}}\right) p_1 + \left(\frac{1}{\sqrt{Re}}\right)^2 p_2 + \dots \end{aligned} \quad (14)$$

Substituting (14) in equations (11) (12) and (13) and then comparing the coefficients of the zeroth power of Re , we get

$$\frac{\partial u_0}{\partial t'} + u_0 \frac{\partial u_0}{\partial x'} + v_0 \frac{\partial u_0}{\partial y'} = - \frac{\partial p_0}{\partial x'} + \frac{\partial^2 u_0}{\partial y'^2}$$

$$0 = -\frac{\partial p_0}{\partial y'} \quad \text{--- (16)}$$

and

$$\frac{\partial u_0}{\partial x'} + \frac{\partial v_0}{\partial y'} = 0 \quad \text{--- (17)}$$

Reverting to the dimensional form and dropping the index zero in (15), (16), (17) we obtain

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad \text{--- (18)}$$

$$\frac{\partial p}{\partial y} = 0 \quad \text{--- (19)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{--- (20)}$$

Now equation (19) shows that p is a function of x only. Hence the pressure is constant in a direction normal to the boundary layer and may be equal to that at the outer edge

of the boundary layer where it is determined by the inviscid flow (Potential flow). Thus we have

$$-\frac{1}{\rho} \frac{dp}{dx} = \frac{\partial \psi}{\partial t} + U \frac{\partial U}{\partial x}$$

where U is the potential flow velocity.

Hence the Prandtl boundary layer equations for a two-dimensional unsteady incompressible flow are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + v \frac{\partial U}{\partial y}$$

$$\text{and } \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad \rightarrow (22)$$

In case of steady flow the above system of equations simplifies to

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$\text{where } -\frac{1}{\rho} \frac{\partial p}{\partial x} = U \frac{\partial U}{\partial x}$$

Now due to no-slip condition and the wall being solid (non-porous), both u and v must vanish at $y=0$.

The boundary conditions are
 $y=0 : u=v=0 ; y \rightarrow \infty : u=U(x)$

Boundary layer flow over a flat plate. Blasius-Topper solution (or simply Blasius Solution) Blasius equation.

Consider a thin infinite flat plate submerged in steady incompressible plane parallel flow, whose undisturbed velocity is U . Since the plate is infinite length, the flow may be regarded as two-dimensional.

Let the origin of the coordinate system coincide with the front edge of the plate, the x -axis

lying along the plate ~~to~~ parallel to
 x and y -axis normal to the plate.

The velocity U of the potential flow is
 constant in this case so that $\frac{dU}{dx} = 0$
 and hence $\frac{dp}{dx} = 0$ $\left[\because \frac{dp}{dx} = -\rho U \frac{dU}{dx} \right]$

Then the Prandtl boundary layer equations
 in the case under consideration are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{--- (2)}$$

The boundary conditions to be satisfied

for u and v are

$$u = v = 0 \text{ when } y = 0, \quad u = U, \quad y = \infty \quad \text{--- (3)}$$

The integration of (1) and (2) can be simplified

by reducing the number of unknowns
 with the help of the stream function
 ψ defined by

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \quad \text{--- (4)}$$

Then (2) is automatically satisfied by (4)

Boundary layer flow over a flat plate. Blasius equation (Blasius-Töpfer solution/Blasius solution)

The order of the boundary ~~thickness~~ layer thickness is $(\nu x / U)^{1/2}$.

Hence we take the new dimensionless distance parameter

$$\eta = \frac{y}{\delta} \quad \text{so that} \quad \eta = y \cdot x^{-1/2} \left(\frac{U}{\nu} \right)^{1/2} \quad (5)$$

In accordance with the ~~produce~~ ^{from (5) previous} of the law of similarity, let the velocity profile be

$$\frac{u}{U} = F(\eta) \quad (6)$$

Using (4), (5) and (6) the stream function ψ is given by

$$\psi = \int u dy = \frac{U}{\sqrt{U\nu x}} \int F(\eta) d\eta = \sqrt{U\nu x} f(\eta)$$

$$= \sqrt{U\nu x} f(\eta)$$

where $f(\eta) = \int F(\eta) d\eta$, then using (4), (5) & (6) we have

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial y} = \sqrt{U\nu x} f'(\eta) \cdot \left(\frac{U}{\nu x} \right)^{1/2} = U f'(\eta)$$

$$\begin{aligned} v &= - \frac{\partial \psi}{\partial x} = - \left[f(\eta) \frac{\partial \sqrt{U\nu x}}{\partial x} + \sqrt{U\nu x} f'(\eta) \frac{\partial \eta}{\partial x} \right] \\ &= - \frac{1}{2} \sqrt{\frac{U\nu}{x}} f(\eta) - \sqrt{U\nu x} f'(\eta) \left(-\frac{1}{2} x^{-3/2} y \sqrt{U} \right) \end{aligned}$$

$$= -\frac{1}{2} \sqrt{\frac{Uv}{x}} f(\eta) + \frac{1}{2} \sqrt{\frac{Uv}{x}} \eta f'(\eta) \quad \text{using (5)}$$

$$\text{Thus } v = \frac{1}{2} \sqrt{\frac{Uv}{x}} [\eta f'(\eta) - f(\eta)] \quad \text{--- (9)}$$

$$\frac{\partial u}{\partial x} = U f''(\eta) \left(-\frac{1}{2} x^{-\frac{3}{2}}\right) \sqrt{\frac{U}{v}}$$

$$\text{--- (2)} \quad -\frac{U}{2x} \sqrt{\frac{U}{v}} f''(\eta) = -\frac{U\eta}{2x} f''(\eta) \quad \text{using (5)} \quad \text{--- (10)}$$

$$\frac{\partial u}{\partial y} = U f'(\eta) \sqrt{\frac{U}{vx}} \quad \text{--- (11)}$$

$$\frac{\partial^2 u}{\partial y^2} = U f'''(\eta) \left(\sqrt{\frac{U}{vx}}\right)^2 = U f'''(\eta) \frac{U}{vx} \quad \text{--- (12)}$$

Substituting (8) to (12) into (1) we get

$$-\frac{U^2}{2x} f' f'' - \frac{U^2}{2x} (\eta f' - f) f'' = \frac{U^2}{x} f'''$$

$$\Rightarrow f f'' + 2 f''' = 0 \quad \text{which is known as the Blasius equation.} \quad \text{--- (13)}$$

Using (5) we see that $y=0 \Rightarrow \eta=0$ and

$y=\infty \Rightarrow \eta=\infty$. Then from (8) & (9) we find

that $u=0, v=0$ at $y=0 \Rightarrow f=0, f'(\eta)=0$, at $\eta=0$

Furthermore (8) shows that $u=U \Rightarrow f'=1$ — (14)

Hence the boundary conditions (3) may be written as $f=0$, $f'=1$ when $\eta=0$ and $f'=1$ when $\eta=\infty$.

Since (13) is a third-order non-linear equation, the three boundary conditions (14) are sufficient to determine the solution completely. But the general solution of (13) has not been possible in closed form. H. Blasius obtained the solution ϕ in the form of a power series expansion about $\eta=0$.

The power series near $\eta=0$ is assumed to be of the form

$$f(\eta) = A_0 + A_1\eta + A_2\frac{\eta^2}{2!} + \frac{A_3}{3!}\eta^3 + \dots \quad (15)$$

$$f'(\eta) = A_1 + A_2\eta + \frac{A_3}{2!}\eta^2 + \frac{A_4}{3!}\eta^3 + \dots \quad (16)$$

$$f''(\eta) = A_2 + A_3\eta + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 + \dots \quad (17)$$

$$f'''(\eta) = A_3 + A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \dots \quad (18)$$

From (14) $f(0) = f'(0) = 0$ at $\eta=0$, hence (15), (16) give

$$A_0 = 0 \quad \text{and} \quad A_1 = 0$$

Substituting (15) and (18) into (13), we get

$$2A_3 + 2A_4\eta + (A_2^2 + 2A_5)\frac{\eta^2}{2!} + (9A_2A_3 + 2A_6)\frac{\eta^3}{3!} + \dots = 0 \quad (19)$$

which is an identity and hence all coefficients of the various powers of η must vanish identically.

Thus we obtain $A_3 = A_4 = A_6 = A_7 = 0$, $A_5 = \frac{A_2^2}{2}$ and $A_8 = \frac{11A_2^2}{4}$. — (20)

With these values (15) reduces to

$$f(\eta) = \frac{A_2}{2!} \eta^2 - \frac{1}{2} \frac{A_2^2}{5!} \eta^5 + \frac{1}{4} \frac{11A_2^3}{8!} \eta^8 - \frac{1}{8} \frac{375A_2^4}{(11)!} \eta^{11} + \dots \quad (21)$$

where the constant A_2 is, for the time being, undetermined, because the third boundary condition at $\eta = \infty$ still remains to be satisfied.

Re-writing (21), we obtain

$$f(\eta) = A_2^{1/3} \left[\frac{(A_2^{1/3} \eta)^2}{2!} - \frac{1}{2} \frac{(A_2^{1/3} \eta)^5}{5!} + \frac{1}{4} \frac{11(A_2^{1/3} \eta)^8}{8!} - \frac{1}{8} \frac{375(A_2^{1/3} \eta)^{11}}{(11)!} \right]$$

$$f(\eta) = A_2^{1/3} F(A_2^{1/3} \eta) \quad \text{say} \quad (21)$$

$$\begin{aligned} \text{Thus } f'(\eta) &= A_2^{1/3} \times F'(A_2^{1/3} \eta) \cdot A_2^{1/3} \\ &= A_2^{2/3} F'(A_2^{1/3} \eta). \end{aligned} \quad (22)$$

from (14), $f'(\eta) = 1$ when $\eta = \infty$. Hence (22) gives

$$\lim_{\eta \rightarrow \infty} A_2^{2/3} F'(A_2^{1/3} \eta) = f'(\infty) = 1 \quad \text{so that } A_2 = \left[\frac{1}{\lim_{\eta \rightarrow \infty} F'(\eta)} \right]^{3/2}$$

The value of A_2 can be obtained numerically from (23) to any desired approximation.

Howarth found that $A_2 = 0.332$.

This completes the solution which is also known as Blasius solution. Using (17), we find

$$f''(0) = A_2 = 0.332.$$

Determination of shearing stress and boundary layer thickness; Using (1).

Using (11) and the facts $y=0 \Rightarrow \eta=0$ and $u=v$ shearing stress τ_0 on the plate is given by

$$\begin{aligned}\tau_0 &= \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} = \mu U f''(0) \left(\frac{U}{\nu x} \right)^{\frac{1}{2}} = \frac{\nu \rho U^2 f''(0)}{(U \nu x)^{\frac{1}{2}}} \\ &= \frac{f''(0) \rho U^2}{(U \nu x)^{\frac{1}{2}}} = \frac{0.332 \rho U^2}{(Re_x)^{\frac{1}{2}}} \quad (24)\end{aligned}$$

Where $Re_x = \frac{Ux}{\nu}$ = Reynold's number

The local skin friction coefficient or friction drag coefficient is given by

$$C_f = \tau_0 / (\rho U^2 / 2) = \frac{0.664}{(Re_x)^{\frac{1}{2}}} \quad \text{using (24)} \quad (25)$$

Raynold's number

(iii)

The Reynold's number Re is defined as

$$\begin{aligned} Re &= \frac{\text{Inertia force}}{\text{Viscous force}} = \frac{\text{Mass} \times \text{Acceleration}}{\text{Shear stress} \times (\text{cross sectional area})} \\ &= \frac{\text{Volume} \times \text{Density} \times (\text{Velocity} / \text{Time})}{\text{Shear stress} \times \text{Cross sectional area}} \\ &= \frac{\text{Cross sectional area} \times \text{Linear dimension} \times \rho \times (\text{velocity} / \text{Time})}{\text{Shear stress} \times \text{Cross sectional area}} \\ &= \frac{(\text{Velocity})^2 \times \rho}{\mu (du/dy)} = \frac{V^2 \rho}{\mu V/L} = \frac{VL}{\nu} \quad \left(\nu = \frac{\mu}{\rho} \right) \end{aligned}$$

Where L and V denote the characteristic length and velocity respectively so that velocity will be proportional to V and du/dy will be proportional to V/L . If for any flow problem Re is small then we can ignore the inertia force, whereas if Re is large then we can neglect the effect of the viscous force and consequently the fluid may be treated as non-viscous fluid.

Froude number

The Froude number Fr is defined by

$$Fr = \frac{\text{Inertia force}}{\text{Gravity force}} = \frac{\text{Mass} \times \text{Acceleration}}{\text{Mass} \times \text{Acceleration due to gravity}}$$

$$= \frac{(\text{velocity})^2}{g \times \text{linear dimension}} = \frac{V^2}{g L}$$

$$= \frac{\text{velocity/time}}{g} = \frac{\text{velocity}}{g \times \text{time}}$$

$$= \frac{\text{velocity}}{g \times (\text{linear dimension/velocity})}$$

$$= \frac{(\text{velocity})^2}{g \times \text{linear dimension}} = \frac{V^2}{g L}$$

Where L and V denote the characteristic length and characteristic velocity respectively so that the velocity will be proportional to V and linear dimension will be proportional to L .

Mach number.

The mach number M or Ma is defined by

$$Ma = M = \frac{V}{c}$$

Where V is the velocity of flow and c is the velocity of sound. Mach number is also expressed in terms of the ratio of inertia force and the elastic force. When the mach number

is small (i.e., $M \ll 1$) the fluid can be taken as incompressible. On the other hand, if Mach number is nearly one or greater than one, the fluid will be taken as compressible.

Prandtl Number.

The Prandtl number is denoted by Pr and defined by

$$Pr = Pr = \frac{\mu C_p}{k}$$

where C_p is the ~~the~~ specific heat at constant pressure and k is the conductivity. Evidently Pr depends only on the properties of the fluid.

Eckert number

The Eckert Ec is defined by

$$Ec = \frac{V^2}{C_p T_0}$$

where T_0 = temperature difference between the wall and the fluid at a large distance from the body

Some dimensionless coefficients employed in the study of flow of viscous fluids

Source and sink is two-dimensional

In two dimensions a source of strength m is such that the flow across any small curve surrounding is $2\pi m$.

Consider a circle of radius r with source at its centre. Then radial velocity q_r is given by

$$q_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{--- (1) or } q_r = -\frac{\partial \phi}{\partial r}$$

$$\text{or } \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{--- (2)}$$

Then the flow across the circle is $2\pi r q_r$. Hence

$$\text{we have } 2\pi r q_r = 2\pi m$$

$$\Rightarrow r q_r = m \quad \text{--- (3)}$$

$$\text{or } r \left(-\frac{1}{r} \frac{\partial \psi}{\partial \theta} \right) = m \quad \text{(by (1)) ---}$$

Integrating and omitting constant of integration, we get.

$$\psi = -m\theta \quad \text{--- (4)}$$

Using (2) & (3) we obtain as before

$$\phi = -m \log r \quad \text{--- (5)}$$

Equation (4) shows that the streamlines are $\theta = \text{constant}$, i.e. straight lines radiating from the source. Again (5) shows that

the curves of equi-velocity potential are $r = \text{constant}$, i.e. concentric circles with centre

of the source.

Complex potential due to source

Let there be a source of strength m at origin. Then

$$w = \phi + i\psi$$

$$= -m \log r - i m \theta$$

$$= -m (\log r + i \log e^{i\theta})$$

$$z = m \log(re^{i\theta})$$

$$z = m \log z$$

If however, the source is at z' , then the complex potential is given by $w = -m \log(z - z')$

The relation between w and z for sources of strengths $m_1, m_2, \dots$, situated at the points $z = z_1, z_2, z_3, \dots$ is given by

$$w = m_1 \log(z - z_1) - m_2 \log(z - z_2) - m_3 \log(z - z_3) - \dots$$

leading to $\phi = -m_1 \log r_1 - m_2 \log r_2 - m_3 \log r_3 - \dots$

and $\psi = -m_1 \theta_1 - m_2 \theta_2 - \dots$

where $r_n = |z - z_n|$, and $\theta_n = \arg(z - z_n)$, $n = 1, 2, 3, \dots$

Example

What arrangement of sources and sinks will give rise to the function $w = \log(z - a/2)$

Draw a rough sketch of the streamlines. Prove that two of the streamlines subdivide into the circle $|z| = a$ and axis of y .

Given

$$w = \log(z - a/2) = \log\left(\frac{(z-a)(z+a)}{z}\right)$$

$$\equiv \log(z-a) + \log(z+a) - \log z$$

This shows that there are two sinks of unit strength at the points $z=a$ and $z=-a$ and a source of unit strength at origin.

Since $w = \phi + i\psi$ and $z = x + iy$, we obtain

$$\phi + i\psi = \log(x+iy-a) + \log(x+iy+a) - \log(x+iy)$$

$$= \log((x-a)+iy) + \log((x+a)+iy) - \log(x+iy)$$

Equating imaginary parts on both sides, we have

$$[\log(\alpha + i\beta)] = \frac{1}{2} \log(\alpha^2 + \beta^2) + i \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

$$\psi = \tan^{-1} \frac{y}{x-a} + \tan^{-1} \frac{y}{x+a} - \tan^{-1} \frac{y}{x}$$

$$= \tan^{-1} \frac{\frac{y}{x-a} + \frac{y}{x+a}}{1 - \frac{y}{x-a} \cdot \frac{y}{x+a}} - \tan^{-1} \frac{y}{x}$$

$$= \tan^{-1} \frac{2xy}{x^2 - y^2 - a^2} - \tan^{-1} \frac{y}{x}$$

$$= \tan^{-1} \frac{\frac{2xy}{x^2+y^2-a^2} - \frac{y}{x}}{1 + \frac{2xy}{x^2+y^2-a^2} \cdot \frac{y}{x}}$$

$$= \tan^{-1} \frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)}$$

The desired streamlines are given by ψ

$$\psi = \text{Constant} = \tan^{-1}(C) \quad \text{i.e.,}$$

$$\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = C$$

When $C=0$, (1) reduces to $y=0$.

Thus x -axis is a streamline.

Again $C \rightarrow \infty$, (1) reduces to $x(x^2+y^2-a^2)=0$

i.e., $x=0$ and $x^2+y^2=a^2$ or $r=a$.

which are streamlines.

Find the stream function of the two-dimensional ^{irrotational} motion due to two equal sources and an equal sink situated midway between them.

→ Let there be two sources of strength m at the points $z=a$ and $z=-a$ and a sink of same strength at $z=0$ (origin). Then complex potential w due to these sources and sink is given by

$$w = -m \log(z-a) - m \log(z+a) + m \log(z-0)$$

$$\phi + i\psi = m \log(x+iy) - m \log(x+ia) - m \log(x+iy+a)$$

$$= m \left\{ \frac{1}{2} \log(x^2+y^2) + i \tan^{-1} \frac{y}{x} \right\} - m \left\{ \frac{1}{2} \log((x-a)^2+y^2) + i \tan^{-1} \frac{y}{x-a} \right\} - m \left\{ \frac{1}{2} \log((x+a)^2+y^2) + i \tan^{-1} \frac{y}{x+a} \right\}$$

$$+ i \tan^{-1} \frac{y}{x} \left\{ -m \left\{ \frac{1}{2} \log((x-a)^2+y^2) + i \tan^{-1} \frac{y}{x-a} \right\} - m \left\{ \frac{1}{2} \log((x+a)^2+y^2) + i \tan^{-1} \frac{y}{x+a} \right\} \right\}$$

Equating imaginary parts on both sides, we get

$$\frac{\psi}{m} = m \tan^{-1} \frac{y}{x} - m \left[\tan^{-1} \frac{y}{x-a} + m \tan^{-1} \frac{y}{x+a} \right]$$

$$\frac{\psi}{m} = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{2xy}{x^2-y^2-a^2}$$

$$\frac{\psi}{m} = \tan^{-1} \frac{(y/x) - \left\{ 2xy/(x^2-y^2-a^2) \right\}}{1 + (y/x) \left\{ 2xy/(x^2-y^2-a^2) \right\}}$$

$$= m \tan^{-1} \frac{y(x^2-y^2-a^2)}{x(a^2-x^2-y^2)}$$

Streamline or line of flow

A streamline is a curve drawn in the ~~position~~ ~~vector~~ of fluid so that its tangent at each point is the direction of motion (i.e., fluid velocity) at that point.

A streamline or line of flow at any instant is a line drawn in the fluid such that the tangent to this line at any point is the direction of the velocity of the fluid at that point.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point P on a streamline and let $\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$ be the fluid velocity at P. Then $\vec{v}$ is parallel to $d\vec{r}$ at P on the streamline.

Thus the equation of streamlines is given by

$$\vec{v} \times d\vec{r} = 0$$

$$(v dz - w dy)\hat{i} + (w dx - u dz)\hat{j} + (u dy - v dx)\hat{k} = 0$$

$$\Rightarrow v dz - w dy = 0, \quad w dx - u dz = 0, \quad u dy - v dx = 0$$

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Path line

A path line is the curve or trajectory along which a particular fluid particle travels during its motion.

The differential equation of a path line is

$$\frac{d\vec{r}}{dt} = \vec{q}$$

$$\Rightarrow \frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w$$

Streamtube. If we draw the streamlines from each point of a closed curve in the fluid, we obtain a tube called the streamtube



The velocity potential or velocity function / Potential flow.

Suppose that the fluid velocity at time t is $\vec{q} = (u, v, w)$

Further suppose that at the considered instant t , there exists a scalar function $\phi(x, y, z, t)$ uniform throughout the entire field of flow and such that

$$-d\phi = udx + vdy + wdz \quad \text{--- (1)}$$

~~such that~~

$$\Rightarrow -\left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz\right) = udx + vdy + wdz \quad \text{--- (2)}$$

Then the expression on the R.H.S of (1) is an ~~exact~~ differential and we have

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z} \quad \text{--- (3)}$$

$$\therefore \quad \vec{q} = -\nabla \phi = -\text{grad } \phi \quad \text{--- (4)}$$

ϕ is called the velocity potential. The negative sign in (4) is a convention. It ensures that the flow takes place from the higher to lower potentials.

The necessary and sufficient condition for (4) to hold is

$$\nabla \times \vec{q} = 0 \quad \text{is} \quad \text{curl } \vec{q} = 0 \quad \text{--- (5)}$$

When (5) holds, the flow is known as the potential kind. Thus for potential flow,

$$\text{curl } \vec{q} = \vec{0}. \quad \text{It is also known as irrotational}$$

For such flow the field of $\vec{q}$ is conservative.

The equation of continuity of an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{--- (6)}$$

Suppose that the fluid moves irrotationally,

Then the velocity potential ϕ exists such that

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z} \quad \text{--- (7)}$$

Using (6), (7) reduces to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

showing that ϕ is harmonic function satisfies the Laplace equation $\nabla^2 \phi = 0$, where

The vorticity vector

Let $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ be the fluid velocity such that $\vec{q} \neq \vec{0}$. Then the vector

$$\Omega = \text{curl } \vec{q}$$

is called the vorticity vector.

Rotational and Irrotational motion

The motion of a fluid is said to be irrotational when the vorticity vector Ω of every fluid particle is zero.

When the vorticity vector is different from zero, the motion is said to be rotational.

Since $\Omega = \text{curl } \vec{q}$ and $\Omega = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\hat{k}$

We conclude that the motion is irrotational if

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

When the motion is irrotational i.e., when $\text{curl } \vec{q} = \vec{0}$ then $\vec{q}$ must be of the form $(\text{grad } \phi)$ for

Some function $\phi(x, y, z)$ because $\text{curl grad } \phi = \vec{0}$

Thus velocity potential exists, whenever the fluid motion is irrotational. Again notice that when velocity potential exists, the motion is irrotational.

$$\vec{q} = i (A x^2 y t) + j (B y^2 z t) + k (C z t^2)$$

where A, B and C are constants.

$$\Rightarrow \text{let } \vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$$

Hence here we have

$$u = A x^2 y t, \quad v = B y^2 z t, \quad w = C z t^2$$

The vorticity components $\Omega_x, \Omega_y, \Omega_z$ are given by

$$\Omega_x = \partial w / \partial y - \partial v / \partial z = 0 - \cancel{Ct^2} B y^2 t$$

$$\Omega_y = \partial u / \partial z - \partial w / \partial x = 0 - 0 = 0$$

$$\Omega_z = \partial v / \partial x - \partial u / \partial y = 0 - A x^2 t$$

8. Test whether the motion specified by $\vec{q} = \frac{k^2(x\hat{j} - y\hat{i})}{x^2 + y^2}$ (k is constant) is a possible motion for an incompressible fluid. If so determine the equation of the streamline

$$\Rightarrow \text{(a) } \vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\text{Then here } u = \frac{-k^2 y}{x^2 + y^2}, \quad v = \frac{k^2 x}{x^2 + y^2}, \quad w = 0 \quad \text{--- (1)}$$

The equation of continuity for an incompressible fluid is

$$\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0 \quad \text{--- (2)}$$

$$\text{from (1) } \therefore \frac{\partial u}{\partial x} = \frac{2k^2 xy}{(x^2 + y^2)^2}, \quad \frac{\partial v}{\partial y} = -\frac{2k^2 xy}{(x^2 + y^2)^2}, \quad \frac{\partial w}{\partial z} = 0$$

Hence (2) is satisfied and so the motion specified by given $\vec{q}$ is possible.

The equation of the streamline are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\therefore \frac{dx}{-ky/(x^2+y^2)} = \frac{dy}{kx/(x^2+y^2)} = \frac{dz}{0} \quad \text{--- (3)}$$

$$dz = 0 \Rightarrow z = c_1 \quad \text{--- (4)}$$

Taking the first two fractions (3) and simplifying, we get

$$dx/-y = \frac{dy}{x} \Rightarrow x^2 + y^2 = c_2 \quad (c_2 = \text{const}) \quad \text{--- (5)}$$

(4) and (5) together give the streamlines.

Now whether the motion is of the potential kind and if so determine the velocity potential

$$\Rightarrow \text{curl } \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-ky}{x^2+y^2} & \frac{kx}{x^2+y^2} & 0 \end{vmatrix}$$

$$= k^2 \left\{ \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2} \right\} \hat{k} = 0 \quad \text{--- (5)}$$

Hence the flow is of the potential kind & we can find velocity potential $\phi(x, y)$ such that $\vec{q} = -\nabla\phi$.

Thus we ~~also~~ have

$$\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2 + y^2} \quad \text{--- (6)}$$

$$\frac{\partial\phi}{\partial y} = -v = -\frac{k^2 x}{x^2 + y^2} \quad \text{--- (7)}$$

$$\frac{\partial\phi}{\partial z} = -w = 0 \quad \text{--- (8)}$$

Equation (8) shows that the velocity potential ϕ is function of x and y only so that $\phi = \phi(x, y)$

Integrating (6) $\phi(x, y) = k^2 \frac{y}{y} \tan\left(\frac{x}{y}\right) + f(y) \quad \text{--- (9)}$

From (7) $\frac{\partial\phi}{\partial y} = f'(y) - \frac{k^2 x}{x^2 + y^2} \quad \text{--- (10)}$

Comparing (9) & (10)

$$f'(y) = 0 \Rightarrow f(y) = \text{constant}$$

$$\therefore \phi(x, y) = k^2 \tan\left(\frac{x}{y}\right)$$

Motion in Two-dimensions

Stream function

Let u and v be the components of velocity in two-dimensional motion. Then the differential equation of lines of flow or streamline is

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow v dx - u dy = 0 \quad \text{--- (1)}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad \text{--- (2)}$$

(2) shows that L.H.S of (1) must be an exact differential. $d\psi$ (say). Thus we have
 $v dx - u dy = d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$

So that $u = -\frac{\partial \psi}{\partial y}$, $v = \frac{\partial \psi}{\partial x}$ --- (3)

This function ψ is known as the stream function.

Then using (1) & (3), the streamlines are given by
 $d\psi = 0$ i.e. by the equation $\psi = c$ where c is an arbitrary constant. Thus the stream function is constant along a streamline.

Irrrotational motion in two dimensions

Let there be an irrotational motion so that the velocity potential ϕ exists such that

$$u = -\partial\phi/\partial x \quad \text{and} \quad v = -\partial\phi/\partial y \quad \text{--- (1)}$$

In two-dimensional flow the stream function ψ always exists such that --- (2)

$$u = -\partial\psi/\partial y, \quad v = \partial\psi/\partial x$$

from (1) and (2)

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \quad \text{and} \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \quad \text{--- (3)}$$

which are all known Cauchy-Riemann equations.

Hence $\phi + i\psi$ is an analytic function of $z = x + iy$

Moreover ϕ and ψ are known as conjugate functions.

Diff' (3) w.r.t x and y respectively, we get

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\psi}{\partial x\partial y} \quad \text{and} \quad \frac{\partial^2\phi}{\partial x\partial y} = -\frac{\partial^2\psi}{\partial x^2}$$

$$\text{Since} \quad \frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\psi}{\partial x\partial y} \quad \text{--- (4)} \quad \frac{\partial^2\phi}{\partial x\partial y} = -\frac{\partial^2\psi}{\partial x^2} \quad \text{--- (5)}$$

Since $\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial^2\psi}{\partial y\partial x}$
adding (5) and (4)

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Similarly $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

$\therefore \phi, \psi$ satisfy Laplace's equations when a two dimensional irrotational motion is considered.

Complex potential

Let $w = \phi + i\psi$ be taken as a function of $x + iy$ i.e. z . Thus suppose that $w = f(z)$ i.e.,

$$\phi + i\psi = f(x + iy) \quad \text{--- (1)}$$

Differentiating (1) w.r.t x and y

$$\frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = f'(x + iy) \quad \text{--- (2)}$$

$$\frac{\partial \phi}{\partial y} + i \frac{\partial \psi}{\partial y} = i f'(x + iy) \quad \text{--- (3)}$$

$$\frac{\partial \phi}{\partial y} + i \frac{\partial \psi}{\partial y} = i \left(\frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} \right)$$

$$\therefore \frac{\partial \phi}{\partial y} = - \frac{\partial \psi}{\partial x}, \quad \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

which are Cauchy-Riemann equations,
then w is an analytic function of z and
 w is known as the complex potential.

Conversely, if w is an analytic function of
 z , then its real part is the velocity
potential and imaginary part is the stream
function of an irrotational two-dimensional
motion.

Example 1: If $\phi = 3xy$ find x and y components
of velocity at $(1,3)$ and $(3,3)$. Determine
the discharge passing between streamlines
passing through these points.

The velocity components u and v in x and
 y directions are given by

$$u = -\frac{\partial \phi}{\partial y} = -3x, \quad v = \frac{\partial \phi}{\partial x} = 3y$$

$\therefore$ Hence the velocity components at $(1,3)$ are

$$u = -3, \quad v = 9.$$

$$\text{at } (3,3), \quad u = -9, \quad v = 9.$$

Now we have

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

$$\Rightarrow \frac{\partial \psi}{\partial y} = -3y, \quad \frac{\partial \psi}{\partial x} = 3x$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$= 3x dx - 3y dy$$

$$\psi = 3\frac{x^2}{2} - \frac{3y^2}{2} + C \quad \left(\text{C constant} \right)$$

∴ charge passing between the streamlines
passing through (1,3) and (3,3)

$$\psi(1,3) - \psi(3,3) = \frac{3}{2}(1-9) + \frac{3}{2}(9-9)$$

$$= \underline{-12} \quad \underline{\text{units}}.$$

Q. If the expression for stream function is described by $\psi = x^3 - 3xy^2$, determine whether flow is rotational or irrotational. If the flow is irrotational, then indicate the correct value of the velocity potential. (a) $\phi = y^3 - 3x^2y$

$$\Rightarrow u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{(b) } \phi = -3x^2y$$

$$u = \frac{\partial \psi}{\partial y} = \underline{3x^2 - 3y^2} - 6xy, \quad v = -\frac{\partial \psi}{\partial x} = \underline{-(3x^2 - 3y^2)}$$

$$\frac{\partial v}{\partial x} = -6x, \quad \frac{\partial u}{\partial y} = -6x$$

A two dimensional flow in xy -plane will be irrotational if the vorticity vector component Ω_z in the z -direction is zero.

$$\text{Here } \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -6x - (-6x) = 0$$

Hence the flow is irrotational.

$$\text{Now } u = -\frac{\partial \phi}{\partial x} \quad \text{and} \quad v = -\frac{\partial \phi}{\partial y}$$

For an irrotational flow Laplace equation in ϕ must be satisfied, i.e., $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$
We now check the validity of each given value of ϕ .

$$\textcircled{a} \quad \phi = y^3 - 3xy \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = -6y, \quad \frac{\partial^2 \phi}{\partial y^2} = 6y$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\textcircled{b} \quad \phi = -3x^2y \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = -6y, \quad \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -6y + 0 \neq 0$$

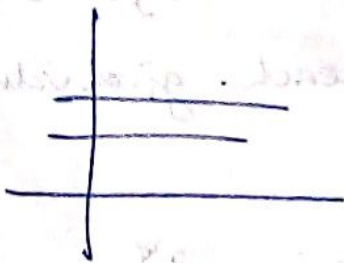
Hence the correct velocity is given by $\phi = y^3 - 3xy$.

Find the stream function ψ for a given velocity potential $\phi = \phi(x)$, where c is a constant. Also, draw a set of streamlines and equipotential lines.

The velocity components u and v in x and y directions are given by

$$u = -\frac{\partial \phi}{\partial x} = -c, \quad v = -\frac{\partial \phi}{\partial y} = 0$$

$$\therefore u = -\frac{\partial \phi}{\partial x} = -c, \quad v = \frac{\partial \psi}{\partial x} = 0$$



$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = c dy$$

$$\psi = cy + d, \quad d \text{ is constant}$$

$$\text{Now } \phi = \text{Constant} \Rightarrow cx = \text{Constant}$$

$$\Rightarrow x = \text{Constant}$$

Showing that the lines of equipotential are parallel to y -axis.

$$\text{Next } \psi = \text{Constant} \Rightarrow cy + d = \text{Constant}$$

$$y = \text{Constant}$$

Showing that the streamlines are parallel to x -axis

1. show that the velocity vector $\vec{q}$ is everywhere tangent to lines in the xy -plane along which $\psi(x, y) = \text{const.}$

→ We have
$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

as
$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \quad (\because \psi = \text{const.} \Rightarrow d\psi = 0)$$

as
$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

∴
$$v dx - u dy = 0$$

$$\frac{dx}{u} = \frac{dy}{v}$$

showing that the velocity vector $\vec{q} = u\hat{i} + v\hat{j}$ is tangent to the streamlines $\psi(x, y) = \text{constant}$.