Fluid :- Fluids are usually classified as liquids and gases.
A liquid has intermolecular forces which hold it together so that it possesses volume but no definite shape. The ability for changes in volume of a mass of fluid is known as compressibility.

Liquids are incompressible, Gas are compressible.

Viscosity :- Viscosity known as internal friction of the fluid.

$$\tau = \mu \frac{du}{dy} \quad \text{(Newton's Law of viscosity)}$$

$$\mu = \frac{\text{shear stress}}{\text{Velocity Gradient}} = \frac{\tau}{du/dy} = \frac{\text{force/area}}{\text{velo/length}} = ML^{-1}T^{-1}$$

$$\nu = \frac{\mu}{S} = \frac{ML^{-1}T^{-1}}{ML^{-3}} = L^{2}T^{-1}$$

Vorticity :- The vector Curl $\bar{V}$ is usually called as 'vorticity vector' or simply vorticity.

$$\bar{\omega} \rightarrow \text{angular velocity} \Rightarrow \bar{\omega} = \frac{1}{2}\,\text{Curl}\,\bar{V}$$

$$\bar{V} = ui + vj + wk$$

$$\omega = (\xi, \eta, \zeta) = \frac{1}{2}\,\text{Curl}\,V = \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k\right]$$

For irrotatimal flow

$$\text{Curl}\,\bar{V} = 0$$

$$\xi = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \eta = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The fundamental equations of the flow of viscous compressible fluids are

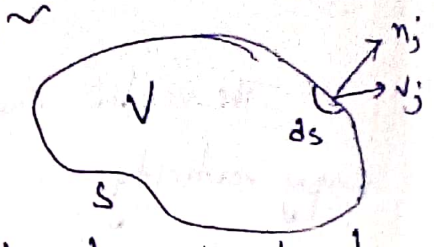
1. Equation of state (one)
2. Equation of continuity (one)
3. Equation of motion (three)
4. Equation of energy (one)

Equation of state :-

$$p = \rho RT$$

For incompressible fluid $\rho = \text{const.}$

Equation of Continuity (Conservation of mass) :-



Let us consider a closed surface S enclosing a fixed (arbitrary) volume V in the region occupied by the moving fluid. If n_j is the unit vector in the direction of the outward drawn normal to the element dS of the surface S and v_j be the ve of the fluid at that point, then inward normal velocity $(-v_j n_j)$.

Thus the mass of the fluid entering, per unit time, through the ds is equal to $\rho(-v_j n_j)ds$

From the above it follows that the mass of the fluid en per unit, time, in the controlled surface S is

$$-\int_S \rho v_j n_j \, ds$$

Mass of the fluid with in S is $\int_V \rho \, dV$

Therefore, the rate of mass increases is $\frac{\partial}{\partial t}\int_V \rho dV$ or $\int_V \frac{\partial \rho}{\partial t} dV$ [∵ V

Conservation of mass

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_S \rho v_j n_j \, ds = -\int_V \frac{\partial}{\partial x_j}(\rho v_j) dV$$ [Gauss

$$\Rightarrow \int_V \left\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j)\right\} dV \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \text{div}(\rho \bar{v}) = 0$$

Equation of Motion (Navier-Stokes' Equations) :-

The equation of motions are derived from Newton's second law of motion which state that

Rate of change of linear momentum = Total force.

Let us consider a closed surface S, enclosing a volume V in the region occupied by moving fluid. The rate at which momentum entering the element dS is $v_i(-\rho\, dS\, v_j n_j)$. Therefore, the rate at which the momentum enters the controlled surface S is

$$-\int_S v_i (\rho v_j n_j)\, dS$$

Also rate at which the momentum increases in the enclosed volume V is

$$\frac{\partial}{\partial t}\int_V \rho v_i\, dV$$

Thus, the rate of change of linear momentum is given by:

$$\frac{\partial}{\partial t}\int_V \rho v_i\, dV + \int_S v_i(\rho v_j n_j)\, dS$$

Therefore the equation of motion

$$\frac{\partial}{\partial t}\int_V \rho v_i\, dV + \int_S v_i(\rho v_j n_j)\, dS = \underbrace{\int_V \rho f_i\, dV}_{\text{Body force}} + \underbrace{\int_S P_i\, dS}_{\text{surdace force}}$$

$$P_i = \sigma_{ij} n_j \quad , \quad \sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$\int_V \left[\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j)\right] dV = \int_V \left[\rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}\right] dV$$

V is arbitrary, thus

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

For isotropic Newtonian fluid, $\tau_{ij} = 2\mu e_{ij} - \frac{2}{3}\mu e_{kk}\delta_{ij}$ $\qquad e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$

Using continuity eqn $\quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0$

$$\rho\left[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right] = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\rho\left[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right] = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}\mu\delta_{ij}\frac{\partial v_k}{\partial x_k}\right]$$

For incompressible fluid $\quad \frac{\partial v_k}{\partial x_k} = 0$

$$\rho\left[\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right] = \rho f_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} \rightarrow \rho\frac{D\bar{v}}{Dt} = \rho\bar{F} - \nabla p + \mu\nabla^2\bar{v}$$

→ N-S eqn for incompressible fluid.

Fundamental equation of a viscous compressible fluid in ordinary Cartesi co-ordinates $x_i = (x, y, z)$ and $v_i = (u, v, w)$

Equation of state $\quad p = \rho R T$

Equation of Continuity $\quad \frac{D\rho}{Dt} + \rho \nabla \cdot \bar{v} = 0$

Eqn of motion in x-direction

$$\rho \frac{Du}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left\{\mu \left(\qquad\qquad\qquad\qquad \right.\right\}$$

Fundamental eqn for incompressible fluid motion

Equation of continuity $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Equation of motion:

x-Component $\quad \rho \frac{Du}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \mu \nabla^2 u$

y- Component $\quad \rho \frac{Dv}{Dt} = \rho f_y - \frac{\partial p}{\partial y} + \mu \nabla^2 v$

z- Component : $\rho \frac{Dw}{Dt} = \rho f_z - \frac{\partial p}{\partial z} + \mu \nabla^2 w$

Equation of Energy : $\quad \rho c_v \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + \kappa \nabla^2 T + \varphi$

$$\varphi = 2\mu\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \mu\left\{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right.\right.$$

$$+ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right.$$

Flow between parallel plates :- (Steady)

$u \neq 0, \quad v = w = 0$

Eqn of continuity

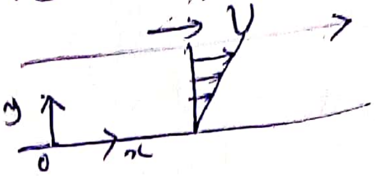
$$\frac{\partial u}{\partial x} = 0$$

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\overset{\text{using cont.}}{\cancel{\frac{\partial^2 u}{\partial x^2}}}^{\,0} + \frac{\partial^2 u}{\partial y^2} + \overset{\text{2-D flow}}{\cancel{\frac{\partial^2 u}{\partial z^2}}}^{\,0}\right)$$

$$0 = -\frac{\partial p}{\partial y}$$

Therefore, $\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} = A \text{ (say)}$

Integrating $u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + Ay + B \rightarrow ①$ A and B are arbitrary constant



Plane Couette flow :-

Boundary conditions

$y: 0 \qquad u = 0$

$y: h \qquad u = U$

Pressure gradient is zero $\frac{\partial p}{\partial x} = 0$

From ① $\quad B = 0$

$$U = \frac{1}{2\mu}\frac{dp}{dx}h^2 + Ah$$

$$\Rightarrow \quad A = \frac{U}{h} - \frac{1}{2\mu}\frac{dp}{dx}h \rightarrow ②$$

from ② & ① $\quad u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + y\left(\frac{U}{h} - \frac{1}{2\mu}\frac{dp}{dx}h\right)$

$$u = Ay + B$$

$$\therefore \frac{u}{U} = \frac{y}{h} \longrightarrow \text{Plane Couette flow.}$$

Plane Poiseuille flow :-

B.C. $\quad y = \pm b \quad : u = 0 \rightarrow ②$

using ② in ①

$$0 = \frac{1}{2\mu}\frac{dp}{dx}b^2 + Ab + B$$

$$0 = \frac{1}{2\mu}\frac{dp}{dx}b^2 - Ab + B$$

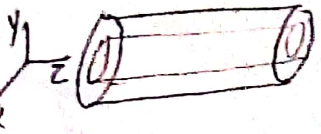
$$B = -\frac{b^2}{2\mu}\frac{dp}{dx} \qquad A = 0$$

$$\therefore u = +\frac{1}{2\mu}\frac{dp}{dx}(y^2 - b^2) = -\frac{1}{2\mu}\frac{dp}{dx}(b^2 - y^2)$$

— The velocity distribution is parabolic $\qquad u_{max} = -\frac{b^2}{2\mu}\frac{dp}{dx}$

$$\therefore \frac{u}{u_{max}} = 1 - \frac{y^2}{b^2}$$

Flow through an Annular section bounded by two concentric cylinders.



we take z-axis as the axis of the cylinder, and the flow is unidirection to the axis. Flow is steady $\frac{\partial}{\partial t}=0$ and $u=v=0$, $w\neq 0$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1)$$

$$\Rightarrow p=p(z)$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (2)$$

$$\cancel{\frac{\partial w}{\partial t}}^{0}+u\cancel{\frac{\partial w}{\partial x}}^{0}+v\cancel{\frac{\partial w}{\partial y}}^{0}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w \, (3)$$

$$w\frac{dw}{dz}=-\frac{1}{\rho}\frac{dp}{dz}+\nu\nabla^2 w \qquad (4)$$

from the equation of continuity $\frac{\partial w}{\partial z}=0$

$$\therefore \nabla^2 w=\frac{1}{\mu}\frac{dp}{dz} \qquad (5)$$

L.H.S of (5) is a function of (x, y), where R.H.S of (5) are the func of z only. This holds only for must equals to constant.

$$\therefore \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{1}{\mu}\frac{dp}{dz}=-P/\mu \text{ (say)} \qquad (6)$$

with boundary condition $w=0$ on the surface of the tube.

The problem can be further simplidied, if we write

$$w=\psi-\frac{P}{4\mu}(x^2+y^2) \longrightarrow (7)$$

Substituting (7) in (6)

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \qquad \text{with B.C. } \psi=\frac{P}{4\mu}(x^2+y^2)$$

In polar co-ordinates

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)=0$$

$$\Rightarrow r\frac{d\psi}{dr}=\text{Const.}=A \text{ (say)}$$

$$\Rightarrow \psi=A\log r+B$$

$r=a$, $\psi=\frac{Pa^2}{4\mu}$ $\qquad \frac{Pa^2}{4\mu}=A\log a+B$

$r=b$, $\psi=\frac{Pb^2}{4\mu}$ $\qquad \frac{Pb^2}{4\mu}=A\log b+B$

$\Rightarrow \psi=A\log\frac{r}{a}+\frac{Pa^2}{4\mu}$

$$A=\frac{P(b^2-a^2)}{4\mu(\log b-\log a)}$$

$$\therefore \psi=\frac{P(b^2-a^2)}{4\mu\log b/a}+\frac{Pa^2}{4\mu}$$

Volume rate flow $Q=\int_0^{2\pi}\int_a^b w\,r\,dr$

$$w=\psi-\frac{P}{4\mu}r^2=\frac{P}{4\mu}\left[\frac{P(b^2-a^2)}{\log b/a}\cdot\log r/a-(r^2-a^2)\right]$$

$$=\frac{\pi P}{8\mu}\left[(a^4-b^4)-\right.$$

Bernoulli's equation :-

Suppose that the body force $\bar{F}(x,y,z)$ acting at any point (x,y,z) of the fluid is such that it can be derived from force potential V.

ie $\bar{F} = -\,\text{grad}\,\bar{V}$ where $\bar{F} = X\bar{i} + Y\bar{j} + Z\bar{k}$

or, $X = -\frac{\partial V}{\partial x}$ $Y = -\frac{\partial V}{\partial y}$ $Z = -\frac{\partial V}{\partial z}$

In this case Euler's equations of motion are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{— (1)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial V}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{— (2)}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \quad \text{— (3)}$$

Equation ① can be written as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{2}(u^2+v^2+w^2)\right] - v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{—④}$$

Let $\frac{1}{\rho} = F(p) \Rightarrow \int \frac{dp}{p} = \int F(p)\,dp \Rightarrow \frac{\partial}{\partial x}\int F(p)\,dp = \left[\frac{\partial}{\partial p}\int F(p)\,dp\right]\frac{\partial p}{\partial x}$

$$= \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Also $\bar{q} = u\bar{i} + v\bar{j} + w\bar{k}$

$$\nabla\times\bar{q} = \bar{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \bar{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \bar{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \bar{i}\,2\xi + \bar{j}\,2\eta + \bar{k}\,2\zeta$$

④ becomes

$$\frac{\partial u}{\partial t} - 2(v\zeta - w\eta) = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}q^2$$

, ρ is a fun^c of p only.

Hence

$$\frac{\partial u}{\partial t} - \bar{q}\times(\nabla\times\bar{q}) = -\frac{\partial H}{\partial x}$$

Euler's eq^n of motion

$$\frac{\partial \bar{q}}{\partial t} - \bar{q}\times(\nabla\times\bar{q}) = -\,\text{grad}\,H$$

(1) for irrotional flow $\nabla\times\bar{q} = 0$ or, $\bar{q} = -\nabla\varphi$, φ is velocity potential

$$-\frac{\partial}{\partial t}(\nabla\varphi) = -\nabla H$$

$$\Rightarrow -\frac{\partial \varphi}{\partial t} + H = \text{Cont.}$$

Therefore $-\frac{\partial \varphi}{\partial t} + \int\frac{dp}{\rho} + \frac{1}{2}q^2 + V = F(t)$ → which is Bernoulli's equation in most general

elocity Potential : Laplace Equation

The two dimensional incompressible continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\text{or, } -\frac{\partial u}{\partial x}=+\frac{\partial v}{\partial y}$$

$$\text{or, } \frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$$

Implies that $-v\,dx+u\,dy=\text{exact}=d\psi$ (say), ψ is known as stream func

$$\text{or, } -v\,dx+u\,dy=\frac{\partial\psi}{\partial y}dy+\frac{\partial\psi}{\partial x}dx$$

$$\Rightarrow u=\frac{\partial\psi}{\partial y} \text{ and } v=-\frac{\partial\psi}{\partial x} \longrightarrow ①$$

Again for irrotational motion

$$\nabla\times\bar{q}=0$$

Implies there exists a scalar potential ϕ, called velocity potential, such that

$$\bar{q}=+\nabla\varphi$$

$$\Rightarrow u=\frac{\partial\varphi}{\partial x} \qquad v=\frac{\partial\varphi}{\partial y} \longrightarrow ②$$

from ① & ②

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial x}&=\frac{\partial\psi}{\partial y}\\ \frac{\partial\varphi}{\partial y}&=-\frac{\partial\psi}{\partial x}\end{aligned}\right\}\text{ Cauchy - Riemann conditions}$$

The stream function and velocity potential satisfy the Laplace equation

$$\nabla^2\varphi=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$$

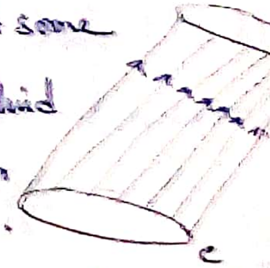
$$\nabla^2\psi=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

Stream Line :-

A stream line or a line of flow at any instant is a line drawn in the fluid such that the tangent to this line at any ~~instant~~ point is the direction of the velocity of the fluid at that point.

Now the direction of motion of a fluid particle at any point (x, y, z) is given by the component of velocity u, v, w therefore the differential equation of the stream lines are $\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$

All stream lines passing through any closed curve C at some time form a tube, which is called a stream tube. No fluid can cross the stream tube because the velocity vector is tangent to the surface.



The path line is the trajectory of a fluid particle of fixed identi over a period of time.

Problem:~ A two-dimensional steady flow has velocity components $u = y$, $v = x$ sho stream lines are rectangular hyperbolas. $x^2 - y^2 =$ Const.

Circulation:~ The circulation Γ around a closed contour C is defined as th line integral of the tangential component of velocity and is given by

$$\Gamma = \oint_c \bar{v} \cdot d\bar{r}$$

Kelvin's Circulation theorem:~

In an inviscid, barotropic flow ~~th~~ with conservative body force, t circulation around a closed curve moving with the fluid remains const. with t

Suppose, we define a complex variable ω whose real and imaginary parts are φ and ψ:

$$\omega = \varphi + i\psi \quad , \; \varphi . \psi \text{ are function of } (x,y)$$

φ, ψ satisfy Cauchy-Riemann Equation.

$$\omega = f(z)$$

$$\frac{d\omega}{dz} = u - iv. \rightarrow \text{Complex velocity}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0. \quad \text{(continuity)}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = 0 \quad \text{(irrotationality)}$$

$$u_r = \frac{\partial \varphi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$u_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

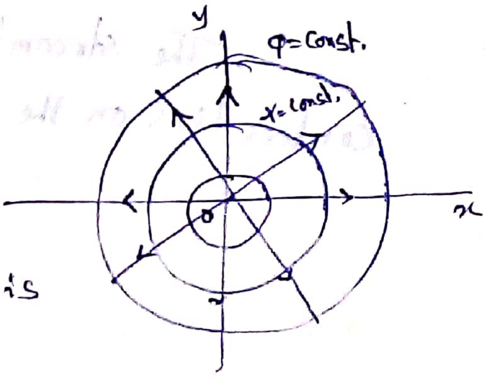
Sources and Sinks :~

Consider the complex potential

$$\omega = \frac{m}{2\pi}\ln z = \frac{m}{2\pi}\ln(re^{i\theta})$$

The real and imaginary parts are

$$\phi = \frac{m}{2\pi}\ln r \qquad \psi = \frac{m}{2\pi}\theta$$



Velocity components are

$$u_r = \frac{m}{2\pi r} \qquad u_\theta = 0$$

For a source situated at z=a, complex potential is

$$\omega = \frac{m}{2\pi}\ln(z-a)$$

Source is a point from which flow emerges from a point in a radially symmetric way.

A source of strength -m is called a sink of strength m. Sink implies absorption of fluid. Example of sink is whirlpool.

Doublet :~

A doublet or dipole is obtained by allowing a source and a sink of equal stre to approach each other in such a way that their strengths increases as the sepa distance goes to zero, and that the product tends to a finite limit.

The complex potential for a source-sink pair on the x-axis a the source at $x = -\epsilon$ and sink at $x = \epsilon$, is

$$\omega = \frac{m}{2\pi}\ln(z+\epsilon) - \frac{m}{2\pi}\ln(z-\epsilon) = \frac{m}{2\pi}\ln\frac{z+\epsilon}{z-\epsilon}$$

$$\approx \frac{m}{2\pi}\ln\left(1+\frac{2\epsilon}{z}+\cdots\right) \approx \frac{m\epsilon}{\pi z}$$

Defining the limit of $m\epsilon/\pi$ as $\epsilon \to 0$ to be μ, the preceding becomes

$$\omega = \frac{\mu}{z} = \frac{\mu}{r}e^{-i\theta}$$

whose real and imaginary parts are

$$\phi = \frac{\mu x}{x^2+y^2} \qquad \psi = -\frac{\mu y}{x^2+y^2}$$

The expression for ψ in the preceding can be rearranged in the form

$$x^2+\left(y+\frac{\mu}{2\psi}\right)^2 = \left(\frac{\mu}{2\psi}\right)^2$$

The streamlines, represented by ψ = const, are therefore circles whos centers lies on the y-axis and are tangents to the x-axis at the origin.

Motion of a Uniform String stream past a Fixed Circular Cylinder.

The combination of a uniform stream and a doublet with its axis directed against the stream gives the irrotational flow over a circular cylinder, for the doublet strength chosen below. The Complex potential for this combination is

$$w = Uz + \frac{\mu}{z} = U\left(z + \frac{a^2}{z}\right) \qquad \mu = a^2 U$$

The real and imaginary parts of w give

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$$

It is seen that $\psi = 0$ at $r = a$ for all values of θ, showing that the stream line $\psi = 0$ represents a circular cylinder of radius a.

Velocity components are

$$u_r = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

From which the flow speed on the surface of the cylinder is found as

$$q\big|_{r=a} = |u_\theta|_{r=a} = 2U\sin\theta$$

This shows that there are stagnation points on the surface, whose polar co-ordinates are $(a,0)$ and (a,π). The flow reaches maximum velocity of $2U$ at top and bottom of the cylinder.

The Pressure distribution on the surface of the cylinder is given by

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = 1 - \frac{q^2}{U^2} = 1 - 4\sin^2\theta \qquad \because p + \frac{1}{2}\rho q^2 = \text{Const} = p_\infty + \frac{1}{2}\rho U^2$$

Flow past a circular cylinder with circulation :-

The complex potential

$$w = U\left(z + \frac{a^2}{z}\right) + \frac{iK}{2\pi} \ln z$$

whose imaginary part is

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{K}{2\pi}\ln(r/a).$$

The tangential velocity component at any point in the flow

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{K}{2\pi r}$$

At the surface of the cylinder, velocity is entirely tangential and given by

$$u_\theta\big|_{r=a} = -2U\sin\theta - \frac{K}{2\pi a}$$

which vanishes if

$$\sin\theta = -\frac{K}{4\pi a U}$$

For $K < 4\pi a U$, two values of θ, satisfies, implying that there are two stagnation points. The stagnation points progressively move down as K increases and coalesce at $K = 4\pi a U$. For $K > 4\pi a U$, the stagn points moves out into the flow along y-axis. The radial distance of stagnation points in this case found from

$$u_\theta\big|_{\theta=-\pi/2} = U\left(1 + \frac{a^2}{r^2}\right) - \frac{K}{2\pi r} = 0$$

This gives

$$r = \frac{1}{4\pi U}\left[K \pm \sqrt{K^2 - (4\pi a U)^2}\right]$$

One root of which is $r > a$, other root corresponds to a stagnation pon inside the cylinder.

The Pressure is found from the Bernoulli' equation

$$p + \rho q^2/2 = p_\infty + \rho\frac{U^2}{2}$$

$$p\big|_{r=a} = p_\infty + \frac{1}{2}\rho\left[U^2 - \left(-2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2\right]$$

The Pressure force along y-axis called the 'lift' force

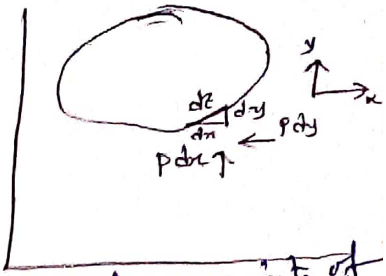
$$L = -\int_0^{2\pi} p\big|_{r=a}\sin\theta \, . \, a\,d\theta = L\rho K$$

lasius Theorem :-

Let a fixed circular cylinder be placed in a liquid which is moving steadily and irrotationally given by the complex potential $\omega = f(z)$. If the hyperdrodynamic pressure on the contour of the cylinder are represented by (X, Y) and the couple about the origin M, then

$$X - iY = \frac{1}{2} i \rho \int_{c} \left(\frac{dw}{dz}\right)^2 dz$$

and $M =$ Real part of $- \frac{1}{2} \int_{c} z \left(\frac{dw}{dz}\right)^2 dz$



Proof :- Consider a general cylindrical body and let X and Y be the x and y components of the force exerted on it by the surrounding fluid.

$$dX = -p\,dy$$
$$dY = p\,dx$$

The total force on the body given by

$$F = X - iY = \int_{c} (-p\,dy) - i \int_{c} p\,dx = -i \int_{c} p(dx - i\,dy) = -i \int_{c} p\,d\bar{z}$$

By Bernoulli's theorem

$$\frac{p}{\rho} + \frac{1}{2} q^2 = \frac{\Pi}{\rho} + \frac{1}{2} U^2$$

, where Π is the pressure at infinity & U is the velocity there

$$p = \left(\Pi + \frac{\rho}{2} U^2\right) - \frac{\rho}{2} q^2$$

$$\therefore \; F = -i\left(\Pi + \frac{\rho}{2} U^2\right) \int_{c} d\bar{z} + \frac{i}{2} \rho \int_{c} q^2 d\bar{z}$$

$$= -i\left(\Pi + \frac{\rho}{2} U^2\right) \int_{c} d\bar{z} + \frac{i}{2} \rho \int_{c} \left(\frac{dw}{dz}\right)^2 dz$$

$$= \frac{i}{2} \rho \int_{c} \left(\frac{dw}{dz}\right)^2 dz$$

$$\int q^2 d\bar{z} = \int \left(\frac{dw}{dz}\right)\left(\frac{d\bar{w}}{d\bar{z}}\right) d\bar{z}$$
$$= \int \left(\frac{dw}{dz}\right) \cdot \frac{d\bar{w}}{d\bar{z}} \cdot d\bar{z}$$
$$= \int \left(\frac{dw}{dz}\right) \cdot d\bar{w}$$
$$= \int \left(\frac{dw}{dz}\right)^2 dz$$

Again $M = \int_{c} p(x\,dx + y\,dy) = \int_{c} p \cdot \text{Real}(z\,d\bar{z}) = \text{Real}\left[\left(\Pi + \frac{\rho}{2} U^2\right) \oint_{c} z\,d\bar{z} - \frac{\rho}{2} \int_{c} q^2 z\,d\bar{z}\right]$

$$= \text{Real}\left[\frac{\rho}{2} \int z \left(\frac{dw}{dz}\right)^2 dz\right]$$