Dirac Equation.

In Schrodinger eq^n, time is of 1st order and space have 2nd order derivative. So we want them to appear in a similar fashion in QFT. However, Klein gordon eq^n has its solution giving negative probability.

With Dirac eq^n, we are going to take the opposite approach and consider only 1st order derivatives for spacial coordinates, keeping the time derivative to 1st order.

The result of Dirac eq^n has the beauty as it successfully able to discribes spin - 1/2 particles.

We know the Schrodinger eq^n as,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 m + V\psi$$

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi} \quad \text{where } \hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V$$

So in order to write dirac eq^n, we change the above Hamiltonian which is applied to wave function. The form of Hamiltonian operator is so chosen that the requirement of Special theory of Relativity is satisfied. We write our Hamiltonian as,

$$\boxed{\hat{H} = c\,\vec{\alpha}\cdot(-i\hbar\nabla) + \beta m c^2}$$

where c = velocity of light
m = rest mass of particle.
α & β is a 4×4 matrices

$\therefore$ Dirac eq^n is written as,

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = \left[c\,\vec{\alpha}\,(i\hbar\vec{\nabla}) + \beta m c^2\right]\psi} \quad \text{—(i)}$$

this is relativistically covariant equation. Here, time and

Space has been put on the same footing as both of them appears in eq^n in terms of their 1^{st} order derivatives.

to rewrite the dirac eq^n interms of most commonly known matrices also known as dirac matrices or gamma matrices, let us 1^{st} Introduce this matrix and its properties

One can write $$\boxed{\vec{\nabla} = \frac{\partial}{\partial x^1}\hat{e}_1 + \frac{\partial}{\partial x^2}\hat{e}_2 + \frac{\partial}{\partial x^3}\hat{e}_3} \quad \text{—(ii)}$$

using this in (i) one can note there is a 1^{st} order spatial derivative appeared explicitly. Now we say α is a vector with it component as 4x4 vectors.

Say, $$\boxed{\vec{\alpha} = \alpha_1\hat{e}_1 + \alpha_2\hat{e}_2 + \alpha_3\hat{e}_3} \quad \text{—(iii)}$$

$\therefore$ gama matries is written as, $$\left.\begin{array}{l} \boxed{\gamma^0 = \beta} \\ \& \ \boxed{\gamma^i = \beta\alpha_i} \end{array}\right\} \text{(iv)}$$

——— x ——— x ——— x ———

Since γ is a matrix, they dont necessarily Commute with each other, ie $\gamma^1\gamma^2 \neq \gamma^2\gamma^1$. In fact they obey the anticommutation rule [~~ie~~. anticommutation ~~tat~~ rule between A & B is given by $\{A, B\} = AB + BA$]

This is given by,

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}} \quad \text{—(v)}$$

Note that the Dirac Matric ie. γ matric corrects them to Space metric (perhaps a connection to quantum gravity here?)

——— x ——— x ——— x ———

let us rewrite dirac eq^n given by (i)

$$i\hbar\frac{\partial\psi}{\partial t} = [c\vec{\alpha}.(+i\hbar\nabla) + \beta mc^2]\psi$$

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So let us work in units where $\hbar = c = 1$.

We have,

$$i\frac{\partial\psi}{\partial t} = [i\vec{\alpha}.\vec{\nabla} + \beta m]\psi$$

multiplying both side by β, from right side.

$$i\left[\beta\frac{\partial\psi}{\partial t} - \beta\vec{\alpha}.\vec{\nabla}\right]\psi = \beta^2 m\psi$$

using all the properties (ii) to (v) one can write

$$i\left[\gamma^0\frac{\partial\psi}{\partial t} - \gamma^i\frac{\partial}{\partial x^i}\right]\psi = \beta(\gamma^0)^2 m\psi$$

$$i\gamma^\mu\partial_\mu\psi = m\psi$$

or

$$\boxed{i\gamma^\mu\partial_\mu\psi - m\psi = 0}$$

$\Downarrow$

this is the most commonly used dirac eqn.

Note: here we have used the properties of γ matrics as $\boxed{(\gamma^\mu)^2 = I}$ which will be later shown as exercise.

This eqn can be applied to dirac fields whose quanta are spin half particles like electrons.

Proof:- Dirac field ψ also satisfy klein-Gordon equation.

It is to note that klein gorden eq^n is nothing but the restatement of Einstein's relation between energy, mass and momentum in special relativity derived using quantum substitution as $E = i\hbar\frac{\partial}{\partial t}$; $p = -i\hbar\frac{\partial}{\partial x}$.

Since $E^2 = p^2+m^2$ (letting $c=\hbar=1$) is an fundamental relation which is applied to all particles and fields, the dirac field ψ must satisfy this condition.

In this sense Dirac eq^n is nothing but 'square root' of K.G. eq^n.

Let us write Dirac eq^n

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0 \quad -(i).$$

operating $i\gamma_\nu\partial^\nu$ from left, we have,

$$-\gamma_\nu\partial^\nu\gamma^\mu\partial_\mu\psi - im\gamma_\nu\partial^\nu\psi = 0$$

or

$$-\gamma_\nu\gamma^\mu\partial^\nu\partial_\mu\psi - im(m\psi) = 0$$

$$\boxed{\because i\gamma_\nu\partial^\nu\psi - m\psi = 0}$$

or

$$\gamma_\nu\gamma^\mu\partial^\nu\partial_\mu\psi + m^2\psi = 0 \qquad -(ii).$$

Here

$$\gamma_\nu\gamma^\mu = g_{\nu\sigma}\gamma^\sigma\gamma^\mu$$

$$= g_{\nu\sigma}\cdot\frac{1}{2}\left[\gamma^\mu\gamma^\sigma + \gamma^\sigma\gamma^\mu\right]$$

using anticommutation relation of γ matri.

i.e. $\{\gamma^\mu,\gamma^\sigma\} = 2g^{\mu\sigma}$

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$$\gamma_\nu \gamma^\mu = g_{\nu\sigma} \cdot \frac{1}{2} [2 g^{\mu\sigma}]$$

$$= g_{\nu\sigma}\, g^{\mu\sigma}$$

$$= \delta_\nu^{\;\mu}$$

∴ we have,

$$\delta_\nu^{\;\mu} \partial^\nu \partial_\mu \psi + m^2 \psi = 0$$

$$\partial^\mu \partial_\mu \psi + m^2 \psi = 0$$

$$\Box \psi + m^2 \psi = 0$$

$$\boxed{(\Box + m^2)\psi = 0}$$

this is nothing but a kleins gorden eqn. hence, dirac field ψ obeys the klein gorden eqn. this mean any dirac field satisfing dirac eqn automatically satisfy the Klein - Gorden relation and hence thereby satisfy the requirements of relativistic mechachics ie relativistic relation between energy man and momentum.

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